Shapes of euclidean polyhedra and hyperbolic geometry

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joint work with François Fillastre (University of Cergy-Pontoise),
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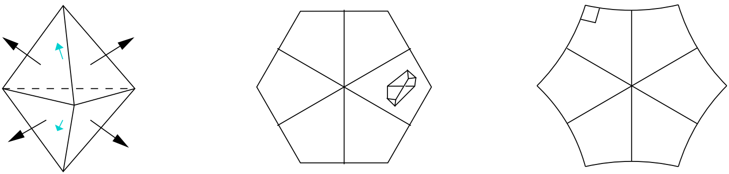
Main result

$V :=$ a (positively spanning) configuration of n unit vectors in $\mathbb{R}^d$.
$M(V) :=$ the space of (homothety classes of) polytopes with outer facet normals V.

Theorem

The space $M(V)$ is a polyhedral ball of dimension $n - d - 1$.
Each cell of $M(V)$ carries a natural hyperbolic metric.



Example

6 vectors in $\mathbb{R}^3$ determine a 2-dimensional complex.
Metrically this is a right-angled hyperbolic hexagon.

Main ingredients

Combinatorics (linear algebra):

studying the combinatorial structure of the polyhedron

$$P(h) := \{Vx \le h\} \subset \mathbb{R}^d, \qquad V \in \mathbb{R}^{n\times d} \text{ fixed}, \quad h \in \mathbb{R}^n \text{ variable}$$

Gale diagrams $\rightsquigarrow h \in C(V)$ a subfan of the secondary fan of V.

Geometry (bilinear algebra):

studying the second intrinsic volume ("quermassintegral")

$$\mathrm{vol}_2(h) = \begin{cases} \mathrm{area}(P(h)) & \text{for } d = 2 \\ \frac{1}{2}\,\mathrm{area}(\partial P(h)) & \text{for } d = 3 \\ \sum_{\dim F = 2} \alpha_F(P(h))\,\mathrm{vol}_2(F) & \text{for } d \ge 4 \end{cases}$$

Alexandrov-Fenchel inequalities for mixed volumes

$\rightsquigarrow$ quadratic form of signature $(+, -, \ldots, -)$.

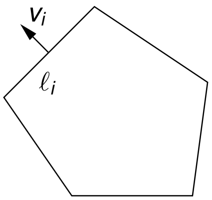
Case $d = 2$

Bavard, Ghys'92:

hyperbolic metric on the space of polygons with fixed edge directions

$\ell_1, \ldots, \ell_n \in C(V) \Leftrightarrow$

$$\sum_{i=1}^{n} \ell_i v_i = 0, \quad \ell_i \geq 0 \, \forall i$$

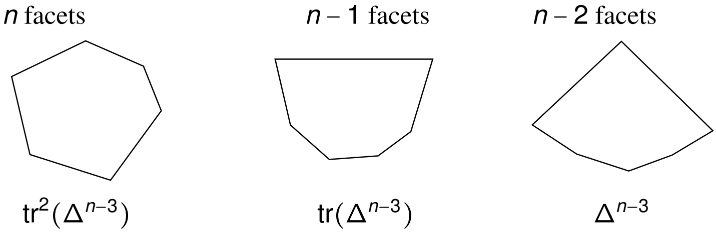


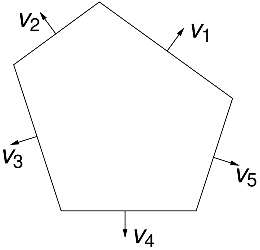
Thus $C(V) = \mathbb{R}^n_{\geq 0} \cap (n-2)$-subspace

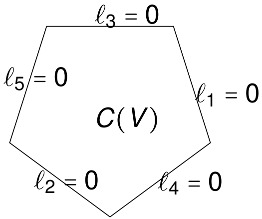
Polygons up to translation = pointed $(n-2)$-cone with $\leq n$ facets.
Polygons up to similarity = $(n-3)$-polytope with $n-2$, $n-1$, or n facets.

Facets of $C(V)$

$\ell_i = 0 \Leftrightarrow i$-th edge contracts to a point







The area is a quadratic form

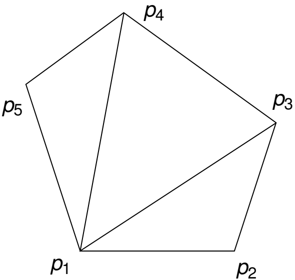
Consider the area of polygons with edge normals V:

$$C(V) \to \mathbb{R} \qquad \ell \mapsto \text{area}(P(\ell))$$

Theorem

$$\text{area}(P(\ell)) = q(\ell)$$

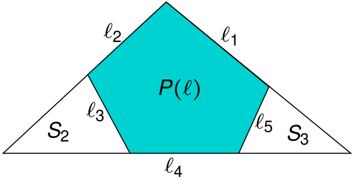
where q is a quadratic form of signature $(+, -, \ldots, -)$.



Why quadratic:
Vertex coordinates are linear functions of ℓ

$$\text{area}(P(\ell)) = \frac{1}{2}\big(\det(p_2 - p_1, p_3 - p_1) + \det(p_3 - p_1, p_4 - p_1) + \det(p_5 - p_1, p_4 - p_1)\big)$$

The signature



$$q(\ell) = \mathrm{area}(S_1) - \mathrm{area}(S_2) - \mathrm{area}(S_3)$$
$$\mathrm{area}(S_1) = (a\ell_2 + b\ell_3)^2 = (a'\ell_1 + b'\ell_5)^2$$
$$\mathrm{area}(S_2) = (c\ell_3)^2, \quad \mathrm{area}(S_3) = (d\ell_5)^2$$

After coordinate change

$$x_0 = a\ell_2 + b\ell_3, \quad x_1 = c\ell_3, \quad x_2 = d\ell_5$$

obtain

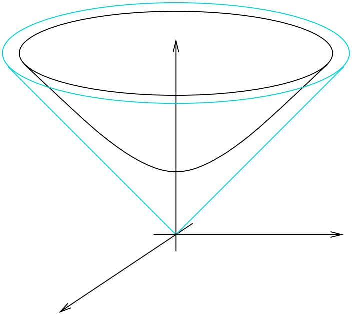
$$\mathrm{area}(P) = x_0^2 - x_1^2 - x_2^2$$

a quadratic form of signature $(+, -, -)$.

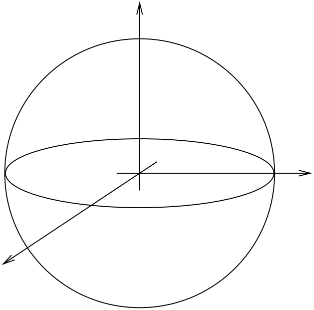
The hyperbolic space

$$q^h(x) = x_0^2 - x_1^2 - \ldots - x_d^2$$
$$\mathbb{H}^d := \{x \mid q(x) = 1, x_0 > 1\}$$

$$q^s(x) = x_0^2 + x_1^2 + \ldots + x_d^2$$
$$\mathbb{S}^d := \{x \mid q(x) = 1\}$$

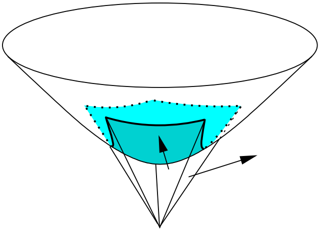


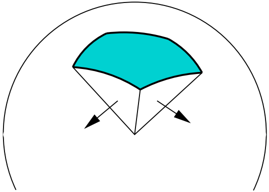
Metric of constant curvature -1.



Metric of constant curvature 1.

Hyperbolic polyhedra





angle of a polygon = dihedral angle of the cone = $\arccos q(\nu_1, \nu_2)$
(in the hyperbolic as in the spherical case)

$$q(\nu_1, \nu_2) = 0 \Leftrightarrow \text{sides form a right angle}$$

If $q(x) = q'(x_0, \ldots, x_{d-2}) - x_{d-1}^2 - x_d^2$, then $\{x_{d-1} = 0\} \perp \{x_d = 0\}$.

Space of polygons is a (truncated) orthoscheme

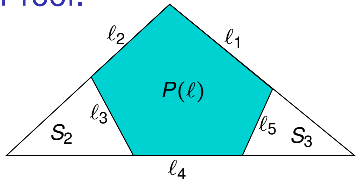
q has signature $(+,-,\ldots,-)$

$$\Rightarrow M(V) := C(V) \cap \{\ell \mid q(\ell) = 1\} \text{ is a hyperbolic polyhedron}$$

Theorem (Bavard,Ghys'92)

If $|i-j| \geq 2$, then $F_i \perp F_j$, where F_i is the facet of $M(V)$ corresponding to contraction of the i-th edge.

Proof.



$$q(\ell) = x_0^2 - x_1^2 - x_2^2$$

$$F_3 = \{x_1 = 0\} \quad F_5 = \{x_2 = 0\}$$

□

Example

The space of equiangular pentagons of area 1 is isometric to the regular right-angled hyperbolic pentagon.

Hyperbolic orthoschemes and complex hyperbolic orbifolds

Bavard, Ghys'92 *Polygones du plan et polyédres hyperboliques*

- Realization of hyperbolic orthoschemes from the Im Hof's list.
- Complete list: Im Hof'90, Tumarkin'07, Kistler'11.

This was motivied by

Thurston'98 *Shapes of polyhedra and triangulations of the sphere*

- Complex hyperbolic structure on the space of Euclidean metrics on $\mathbb{S}^2$ with cone angles $\alpha_1, \ldots, \alpha_n$.
- Realization of some non-arithmetic complex Coxeter orbifolds.

We present a generalization of the Bavard-Ghys construction to higher dimensions.

From $d = 2$ to $d > 2$

Combinatorics

- facet normals V no more determine the combinatorial type of the polytope P
- $\Rightarrow C(V)$ no more a cone, but a fan (made of type cones)

Geometry

- What to use instead of area(P)?
- Is there a canonical quadratic form of signature $(+, -, \ldots, -)$?
- Can the dihedral angles of $M(V)$ be computed?

Type cones

$V = (v_1, \dots, v_n)$ a positively spanning vector configuration in $\mathbb{R}^d$
$C(V) :=$ translation classes of polytopes with facet normals V
$P, P' \in C(V)$ are normally equivalent if they have the same normal fan: $\mathcal{N}(P) = \mathcal{N}(P')$

$$C(V) = \bigsqcup_{\Delta} T(\Delta),$$

where $T(\Delta) = \{P \mid \mathcal{N}(P) = \Delta\}/\{\text{translations}\}$.

Theorem

The closure of $C(V)$ is a pointed fan with convex support.

Example

For $V = \{\pm e_1, \dots, \pm e_d\}$ all P are normally equivalent.

Monotypic polytopes: McMullen, Schneider, Shephard'74.

Support numbers and Gale dual

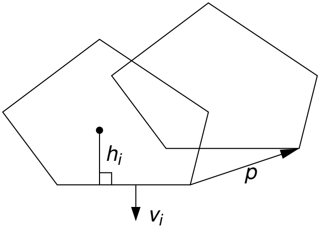
Given $V \in \mathbb{R}^{n \times d}$ with rows $v_i \in \mathbb{R}^d$ of norm 1, consider

$$P(h) = \{x \in \mathbb{R}^n \mid Vx \leq h\}, \quad h \in \mathbb{R}^n$$

Support numbers $(h_i)_{i=1}^n$. We have

$$P(h) + p = P(h + Vp)$$

Hence $C(V) \subset \mathbb{R}^n / \operatorname{im} V$.



Definition $\pi \colon \mathbb{R}^n \to \mathbb{R}^n / \operatorname{im} V, \quad \pi(e_i) =: \bar{v}_i$

Vector configuration $\overline{V} = (\bar{v}_1, \ldots, \bar{v}_n)$ *is called* ***Gale dual*** *to* V.

$$V^\top \overline{V} = E_n, \quad \operatorname{rank} V + \operatorname{rank} \overline{V} = n$$

V positively spanning $\Leftrightarrow$ $\overline{V}$ lies in an open half-space

$C(V)$ lies in $\mathrm{pos}(\overline{V})$

Theorem

$$P(h) \neq \varnothing \Leftrightarrow \pi(h) \in \mathrm{pos}(\overline{V}) := \{\sum_i \lambda_i \bar{v}_i \mid \lambda_i \geq 0\}$$

Proof.

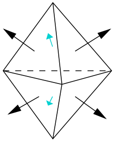
Note: $0 \in P(h) \Leftrightarrow h_i \geq 0 \forall i$.

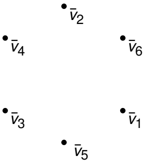
$$P(h) \neq \varnothing \Leftrightarrow \exists p \in P(h) \Leftrightarrow 0 \in P(h) - p = P(h - Vp)$$
$$\Leftrightarrow h - Vp \in \mathbb{R}^n_{\geq 0} \Leftrightarrow \pi(h) \in \mathrm{pos}(\bar{v}_1, \ldots, \bar{v}_n)$$

□

Example

V are normals of the triangular bipyramid $\Rightarrow$ $\overline{V}$ span a hexagonal cone





$C(V)$ is the 2-core of $\overline{V}$

The *k-core* of a vector configuration $W \subset \mathbb{R}^m$ is

$$\mathrm{core}_k(W) = \{x \in \mathbb{R}^m \mid \forall y \text{ s.t. } \langle y, x\rangle \geq 0 \ \exists w_{i_1}, \ldots, w_{i_k} \text{ s.t. } \langle y, w_{i_\alpha}\rangle \geq 0\}$$

E.g. $\mathrm{core}_1(W) = \mathrm{pos}(W)$ $\qquad \mathrm{core}_2(W) = \bigcap_{w \in W} \mathrm{pos}(W \setminus \{w\})$

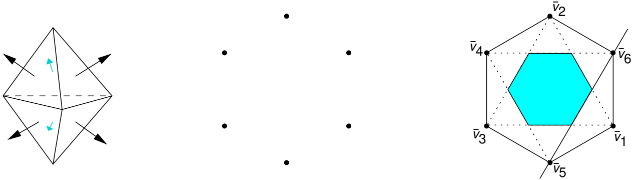
Theorem

The closure of $C(V)$ *is* $\mathrm{core}_2(\overline{V})$.

Proof.

i-th facet non-empty $\Leftrightarrow \exists p \in \mathbb{R}^n$ such that $\langle v_i, p\rangle = h_i$ and $\langle v_j, p\rangle < h_j$ for all $j \neq i$. Then use $P(h - Vp) = P(h) - p$ etc. □

Example



Type cones are the chambers of the chamber fan

The chamber fan $\mathrm{Ch}(W)$ of W is the coarsest common subdivision of all cones spanned by W.

Theorem

Closures of type cones form the chamber fan of $\overline{V}$.

Lemma

$\mathrm{pos}(V_\sigma) \in \mathcal{N}(P(h)) \Leftrightarrow \pi(h) \in \mathrm{pos}(\overline{V}_{\bar{\sigma}})$, *where* $\bar{\sigma} = \{1, \dots, n\} \setminus \sigma$.

Proof.

Similar to the preceding two arguments. □

Example

References

All the above arguments on type cones and their arrangement appeared in

- McMullen'73 *Representations of polytopes and polyhedral sets*
- Shephard'71 *Spherical complexes and radial projections of polytopes*

From this, there is only one step to the secondary polyhedron.

Relation to the secondary polyhedron

Let $P \in C(V)$, $\mathcal{N}(P) = \Delta$. Then its support function

$$h_P : \mathbb{R}^d \to \mathbb{R}$$

is a convex conewise (wrt Δ) linear function such that $h(v_i) = h_i$. Thus

$$\begin{aligned} \text{polytopal fans} &\leftrightarrow \text{regular triangulations} \\ \text{support numbers} &\leftrightarrow \text{weights} \end{aligned}$$

Why is the chamber fan polytopal? Invert the construction:

- V is the Gale dual of $\overline{V}$
- for every polytopal fan Δ on V pick h^Δ such that $P(h^\Delta) \in T(\Delta)$
- then $\mathcal{N}(\sum_\Delta P(h^\Delta)) = \mathrm{Ch}(V)$

The fan $\mathrm{Ch}(V)$ is the normal fan of the secondary polyhedron of $\overline{V}$, and vice versa.

Intrinsic volumes

$K \subset \mathbb{R}^d$ convex body, $B \subset \mathbb{R}^d$ unit ball

$$\mathrm{vol}_d(K + tB) = \sum_{i=0}^{d} W_i(K) \binom{d}{i} t^i$$

The coefficient $W_i(K)$ is called the i-th quermassintegral; it is the mixed volume

$$W_i(K) = \mathrm{vol}_d(K, \ldots, K, B, \ldots, B)$$

The i-th quermassintegral is proportional to the $(d - i)$-th intrinsic volume. If $K = P$ is a polytope, then

$$\mathrm{vol}_{d-i}(P) = \sum_{\dim F = d-i} |N_F(P)| \mathrm{vol}_{d-i}(F),$$

where $|N_F(P)|$ is the normalized exterior angle at F.

Example

$\mathrm{vol}_{d-1}(P) = \frac{1}{2}\mathrm{vol}_{d-1}(\partial P), \quad \mathrm{vol}_0(P) = 1$

The second intrinsic volume is a quadratic form

- mixed volumes are multilinear wrt Minkowski addition and multiplication with positive scalars
- on every $T(\Delta)$, Minkowski addition corresponds to addition of the support numbers

Hence for every Δ there is a quadratic form q_Δ on $\mathbb{R}^n$ s.t.

$$q_\Delta(h) = vol_2(P(h)) = c\,\mathrm{vol}_d(P(h), P(h), B, \ldots, B) \quad \text{if } \mathcal{N}(P(h)) = \Delta$$

More generally,

$$q_{\Delta,K} := \mathrm{vol}_d(P(h), P(h), K_1, \ldots, K_{d-2})$$

The signature of q_Δ

Theorem (Alexandrov-Fenchel inequalities)

$$q_\Delta(h, h')^2 \geq q_\Delta(h, h) q_\Delta(h', h')$$

Corollary

The form q_Δ has exactly one positive eigenvalue.

Proof.

$$q_\Delta(h, h) = \mathrm{vol}_2(P(h)) > 0, \quad \det \begin{pmatrix} q_\Delta(h, h) & q_\Delta(h, h') \\ q_\Delta(h', h) & q_\Delta(h', h') \end{pmatrix} \leq 0$$

$\Rightarrow$ q_Δ is indefinite on all 2-dimensional subspaces through h □

The equality case is known $\Rightarrow$ the form has maximum possible rank.

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The equality case is known $\Rightarrow$ the form has maximum possible rank. (For general $q_{\Delta,K}(h)$ the rank is unknown, since no characterization of the equality case in the AF-inequality.)

Right angles on the boundary of $M(V)$

Each facet of $C(V)$ corresponds either to face truncation or to degeneration of $P(h)$ (dimension falls).

Theorem

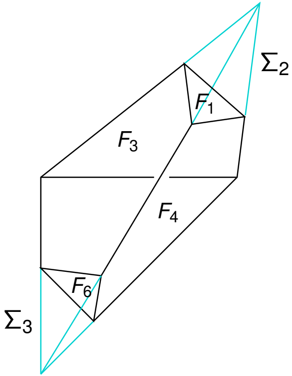
Facets corresponding to truncation of different vertices are orthogonal to each other.

Proof.

$$\text{area}(\partial P(h)) = f_1^2 - f_2^2 - f_3^2$$

$$f_1^2 = \text{area}(\partial \Sigma_1) = (ah_2 + bh_3 + ch_4 + dh_5)^2$$

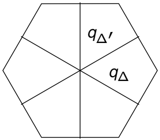
$$\begin{aligned} f_3^2 &= \text{area}(F_3(\Sigma_3)) + \text{area}(F_4(\Sigma_3)) \\ &\quad + \text{area}(F_5(\Sigma_3)) - \text{area}(F_6(\Sigma_3)) \\ &= (a'h_3 + b'h_4 + c'h_5 + d'h_6)^2 \end{aligned}$$



□

From bipyramid to a right-angled hyperbolic hexagon

A priori $q_\Delta(h) \neq q'_\Delta(h)$ and gluing 6 hyperbolic quadrilaterals yields a cone point in the center.



Theorem

In this case $q_\Delta = q_{\Delta'}$. Thus have a right-angled hyperbolic hexagon.

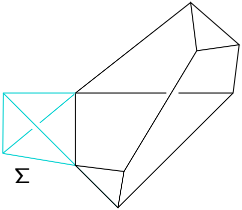
Proof.

This is because

$$q_{\Delta'}(h) = q_\Delta(h) + \text{area}(F_3(\Sigma)) + \text{area}(F_5(\Sigma)) - \text{area}(F_2(\Sigma)) - \text{area}(F_6(\Sigma))$$

and $\text{area}(F_3) + \text{area}(F_5) = \text{area}(F_2) + \text{area}(F_6)$ because

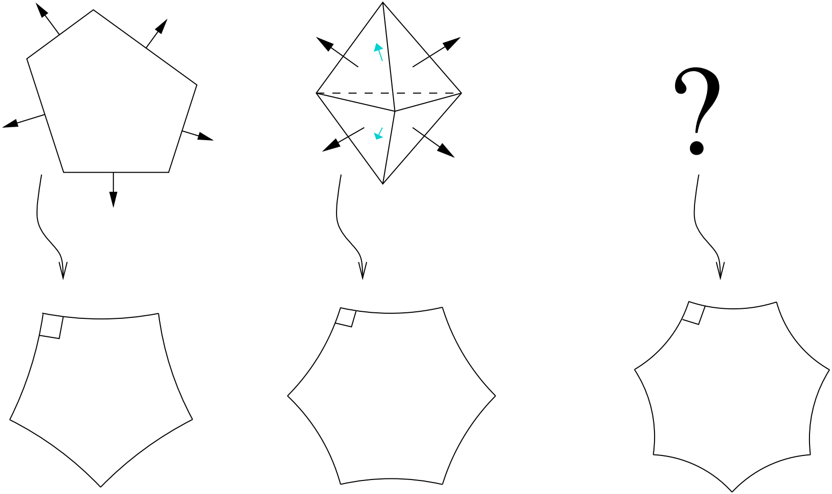
$$v_3 + v_5 = v_2 + v_6$$



□

A riddle

Continue the sequence.



Is “**?**” the polar dual of a cyclic 4-polytope on 7 vertices?