On the covering radius of lattice polytopes and its relation to view-obstructions and densities of lattice arrangements

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based on joint work with

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Definition

For a convex body K in $\mathbb{R}^n$ and a lattice $\Lambda = A\mathbb{Z}^n$, $A \in \mathrm{GL}_n(\mathbb{R})$, we say that

$$K + \Lambda = \bigcup_{z \in \Lambda} (K + z)$$

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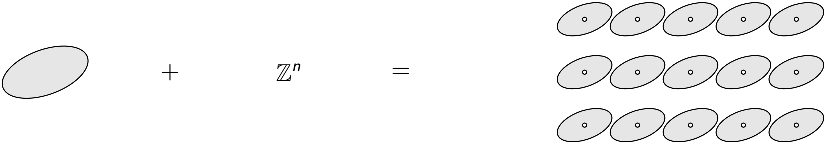

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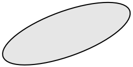

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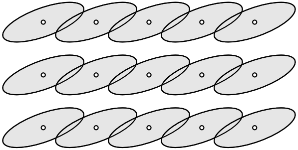

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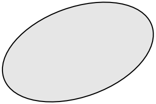

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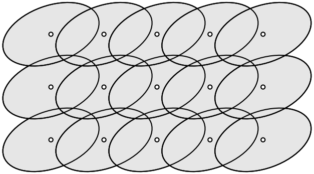

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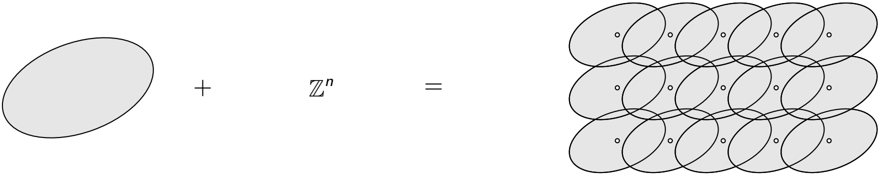

Definition

The lattice of translates $K + \Lambda$ is a *lattice covering* if $K + \Lambda = \mathbb{R}^n$.

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The *covering radius* of $K \subseteq \mathbb{R}^n$ with respect to a lattice Λ is defined as

$$\mu(K, \Lambda) = \min\{\mu > 0 : \mu K + \Lambda = \mathbb{R}^n\}.$$

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Appearances in the literature:

- Coin Exchange Problem of Frobenius (Kannan '92)
- Transference Theorems, Diophantine Approximation (Kannan & Lovász '88)
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Computationally difficult parameter:

- Kannan '93: Polynomial-time algorithm to compute $\mu(P, \Lambda)$ for rational polytopes P in fixed dimension; triple-exponential in the dimension.
- Haviv & Regev '06: It is Π_2-hard to approximate $\mu(B_p^n, \Lambda)$ to within a factor $c_p > 0$ for all sufficiently large $p \geq 1$.
- (Conjecture) Deciding $\mu(B_2^n, \Lambda) \leq \mu$ is NP-hard. (Guruswami et al. '05)

Definition (Kannan & Lovász '88; G. Fejes Tóth '76)

The *i*th *covering minimum* of $K \subseteq \mathbb{R}^n$ with respect to a lattice Λ is defined as

$$\mu_i(K,\Lambda) = \min\{\mu > 0 : \mu K + \Lambda \text{ intersects every } (n-i)\text{-dim. affine subspace}\}.$$

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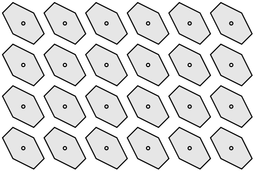

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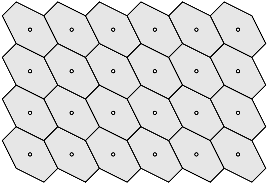

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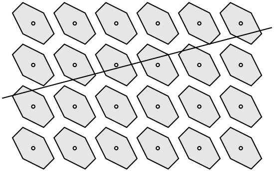

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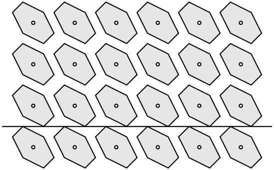

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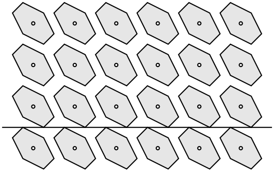

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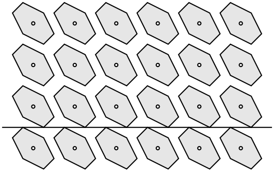

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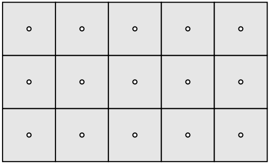
Lemma (Kannan & Lovász '88)

$$\mu_i(K,\Lambda) = \max\{\mu(K|L,\Lambda|L) : L \text{ an } i\text{-dimensional subspace}\}$$

Examples

- For $C_n = [-\frac{1}{2}, \frac{1}{2}]^n$, we have

  $$\mu_i(C_n) = 1 \text{ for each } i = 1, \ldots, n.$$



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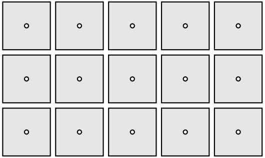

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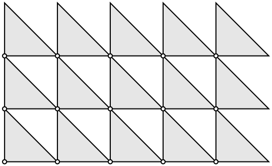

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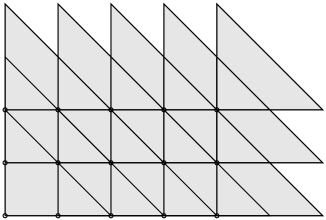

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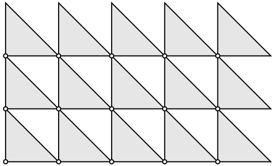

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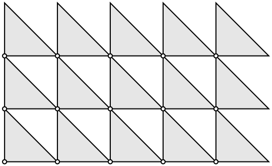
- For the Euclidean unit ball B_2^n, we have

$$\mu_i(B_2^n) = \frac{\sqrt{i}}{2} \text{ for each } i = 1, \ldots, n.$$



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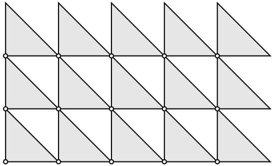

Proposition

Let $P \subseteq \mathbb{R}^n$ be a lattice polytope. Then

- $\mu_i(P) \leq i$*, for every* $i = 1, \ldots, n$*, and*
- *if P is a lattice zonotope, then* $\mu_i(P) \leq 1$*, for every* $i = 1, \ldots, n$*.*

We discuss two problems in which the computation / estimation of covering radii of lattice polytopes plays a crucial role:

❶
Towards a Covering Analog of Minkowski's 2nd Theorem

❷
Rationally Constrained View-Obstruction Problem

Covering analog of Minkowski's 2nd Theorem

Theorem (Minkowski 1896)

For every convex body K in $\mathbb{R}^n$ with $K = -K$, we have

$$\frac{2^n}{n!} \leq \lambda_1(K) \cdot \ldots \cdot \lambda_n(K) \operatorname{vol}(K) \leq 2^n,$$

where $\lambda_i(K) = \min\{\lambda > 0 : \dim(\lambda K \cap \mathbb{Z}^n) \geq i\}$ is the ith successive minimum of K.

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Problem: Find best possible lower bound on $\mu_1(K) \cdot \ldots \cdot \mu_n(K) \operatorname{vol}(K)$, for K in $\mathbb{R}^n$.

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Theorem (Schnell '95)

For every planar convex body K, we have $\mu_1(K)\mu_2(K) \operatorname{vol}(K) \geq \frac{3}{4}$.

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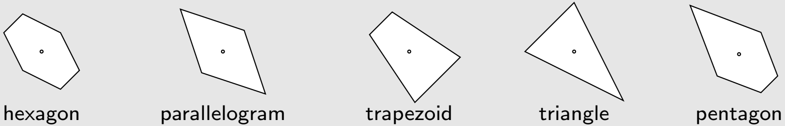
Equality holds if and only if K is lattice-equivalent to one of the following:



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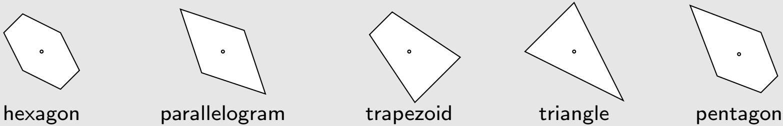

$\rightarrow$ Analogous to *lattice tiles*, that is, K such that $K + \mathbb{Z}^n$ is a covering and a packing.

Covering analog of Minkowski's 2nd Theorem

Theorem (González Merino & H. '16)

i) *For every convex body K in $\mathbb{R}^n$, we have*

$$\mu_1(K) \cdot \ldots \cdot \mu_n(K) \operatorname{vol}(K) \geq \frac{1}{n!}.$$

ii) *For every convex body K in $\mathbb{R}^n$ that is symmetric with respect to every coordinate hyperplane, we have*

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Equality holds for example for the cube $C_n = [-\frac{1}{2}, \frac{1}{2}]^n$.

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Conjecture

For every convex body K in $\mathbb{R}^n$, we have

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$\rightarrow$ extremal example should be $T_n = \mathrm{conv}\{e_1, \ldots, e_n, -\mathbf{1}\}$

Proposition

Let $T_n = \mathrm{conv}\{e_1, \dots, e_n, -\mathbf{1}\}$. *Then*

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Covering Minima of T_n

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Let $T_n = \mathrm{conv}\{e_1, \ldots, e_n, -\mathbf{1}\}$. Then

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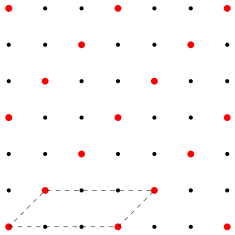

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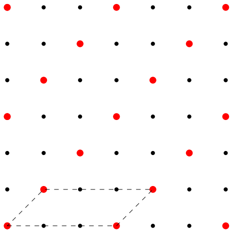

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- vertex set $\mathbb{Z}^n$
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Diameters of Quotient Lattice Graphs

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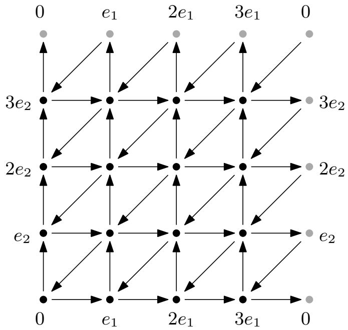

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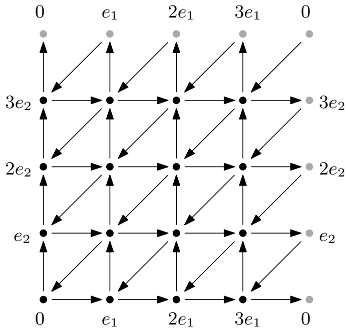

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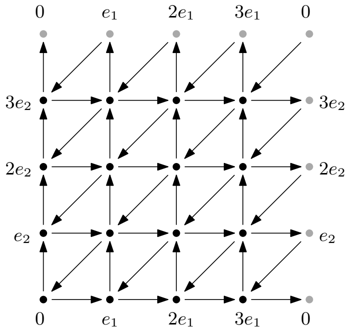

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- averaging argument + elementary number theory □

Open Problems

Problem 1

Prove or disprove an exponential lower bound on the covering product. More precisely, find some $0 < c < 1$ such that

$$\mu_1(K) \cdot \ldots \cdot \mu_n(K) \operatorname{vol}(K) \geq c^n,$$

for every convex body K in $\mathbb{R}^n$.

Problem 2

Find a method to show that $\mu_i(T_n) = \frac{i}{2}$, for $1 \leq i \leq n$.

Problem 3

Extend the approach of Marklof & Strömbergsson to the computation of $\mu_i(S_1, \Lambda)$, $1 \leq i \leq n$, for sublattices $\Lambda \subseteq \mathbb{Z}^n$ via generalized diameters of quotient lattice graphs.

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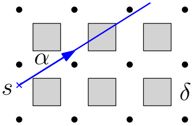

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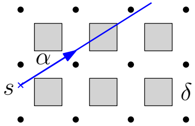

Billiard Ball Motions: (Schoenberg '76)

For $s \in [0,1]^n$ and $\alpha \in \mathbb{R}^n$, let $\text{bbm}(s, \alpha) \subseteq [0,1]^n$ be the trajectory of the motion starting with $s + \lambda\alpha$, $\lambda \geq 0$, and which is reflected naturally in the boundary of the cube $[0,1]^n$.

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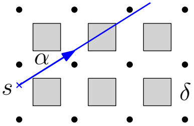

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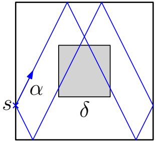

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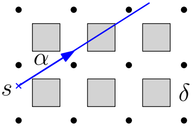

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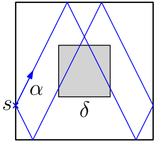

- $\mathrm{view}(s, \alpha)$ is δ-obstructed $\iff$ $\mathrm{bbm}(s, \alpha)$ is δ-central

Schoenberg's Theorem

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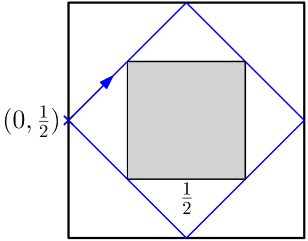

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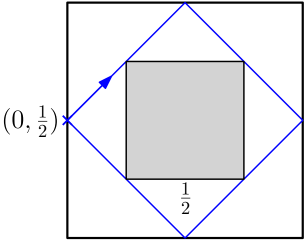

If $\{\alpha_1, \ldots, \alpha_n\}$ is linearly independent over $\mathbb{Q}$, then $\mathrm{bbm}(s, \alpha)$ is dense in $[0,1]^n$.

Definition

The *rational dimension* of $\alpha \in \mathbb{R}^n$ is defined by $\dim_{\mathbb{Q}}(\alpha) = \dim(\mathrm{span}_{\mathbb{Q}}\{\alpha_1, \ldots, \alpha_n\})$.

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A vector $\alpha \in \mathbb{R}^n$ is called *rationally uniform* if every $\dim_{\mathbb{Q}}(\alpha)$ coordinates of α are linearly independent over $\mathbb{Q}$.

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Rationally Uniform Directions

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For every $n \in \mathbb{N}$, we have $0 = \delta(0, n) \leq \delta(1, n) \leq \ldots \leq \delta(n-1, n) = (n-1)/n$.

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For $\alpha \in \mathbb{R}^n$ define

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Theorem (H. & Malikiosis '16)

Let $\delta \geq 0$, let $s, \alpha \in \mathbb{R}^n$, and let $d = n - \dim_{\mathbb{Q}}(\alpha)$. Then,

$$\mathrm{view}(s, \alpha) \text{ is } \delta\text{-obstructed} \quad \iff \quad (\delta Z_\alpha + \bar{s}) \cap \mathbb{Z}^d \neq \emptyset.$$

The covering radius of a convex body $K \subseteq \mathbb{R}^d$ is equivalently given by

$$\mu(K) = \min\{\mu \geq 0 : (\mu K + t) \cap \mathbb{Z}^d \neq \emptyset,\ \forall t \in \mathbb{R}^d\} = \min\{\mu \geq 0 : \mu K + \mathbb{Z}^d = \mathbb{R}^d\}.$$

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α is rationally uniform $\quad \Longleftrightarrow \quad$ Z_α is a cubical lattice zonotope

Consequently,

$$\begin{aligned}\delta(k, n) &= \inf\{\delta \geq 0 : \text{every rationally uniform view}(s, \alpha) \subseteq \mathbb{R}^n \\ &\qquad \text{with } \dim_{\mathbb{Q}}(\alpha) \geq n - k \text{ is } \delta\text{-obstructed}\} \\ &= \sup\{\mu(Z) : Z \subseteq \mathbb{R}^d \text{ a cubical lattice zonotope with } n \text{ generators}, d \leq k\}.\end{aligned}$$

Theorem (Flatness Theorem for Zonotopes; Banaszczyk '96)

Let $Z \subseteq \mathbb{R}^d$ *be a zonotope such that* $\mathrm{int}(Z + t) \cap \mathbb{Z}^d = \emptyset$*, for some* $t \in \mathbb{R}^d$*. Then there exists* $v \in \mathbb{Z}^d \setminus \{0\}$ *and an absolute constant* $c > 0$ *such that*

$$w(Z, v) := \max_{x \in Z}(x^\intercal v) - \min_{x \in Z}(x^\intercal v) \leq c\, d \log d.$$

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Conjecture

For every cubical lattice zonotope $Z \subseteq \mathbb{R}^k$ with n generators holds $\mu(Z) \leq \frac{k}{n}$.
(True for $k \in \{1, n-1, n\}$.)

Open Problems

Problem 1

Find a zonotopal proof of Schoenberg's Theorem, that is, $\mu(Z) \leq \frac{n}{n+1}$, for every cubical lattice zonotope $Z \subseteq \mathbb{R}^n$ with $n+1$ generators.

Problem 2

Identify examples of cubical lattice zonotopes in $\mathbb{R}^k$ with n generators and $\mu(Z) = \frac{k}{n}$.

Problem 3

Is there a theory to relate the covering radius of lattice parallelepipeds to certain graph parameters of quotient lattice graphs (analogous to Marklof & Strömbergsson for lattice simplices)?

Some literature

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Thank you very much!