Einstein Workshop on
Lattice Polytopes

Berlin, December 11th-15th, 2016

Exploiting Symmetries of Lattice Polytopes

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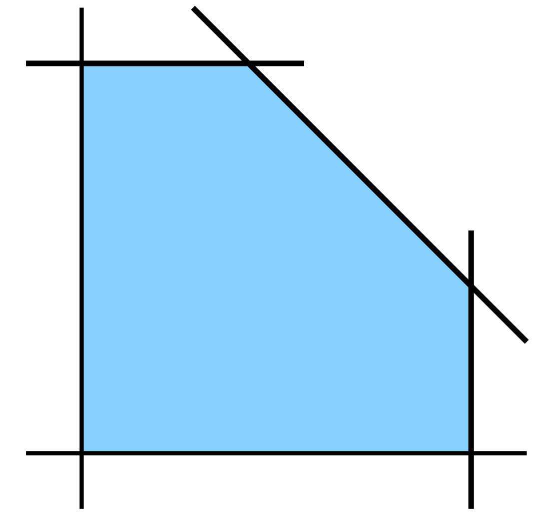
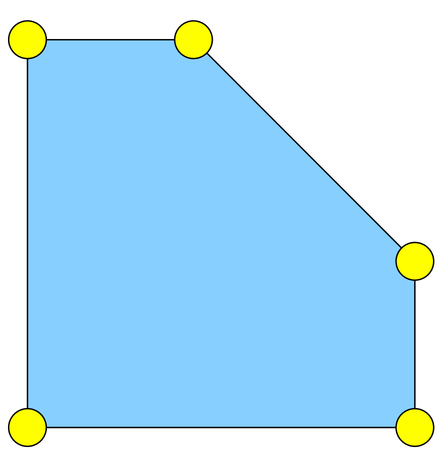
(with parts based on work with David Bremner, Mathieu Dutour Sikirić,
Erik Friese, Katrin Herr, Dima Pasechnik and Thomas Rehn)

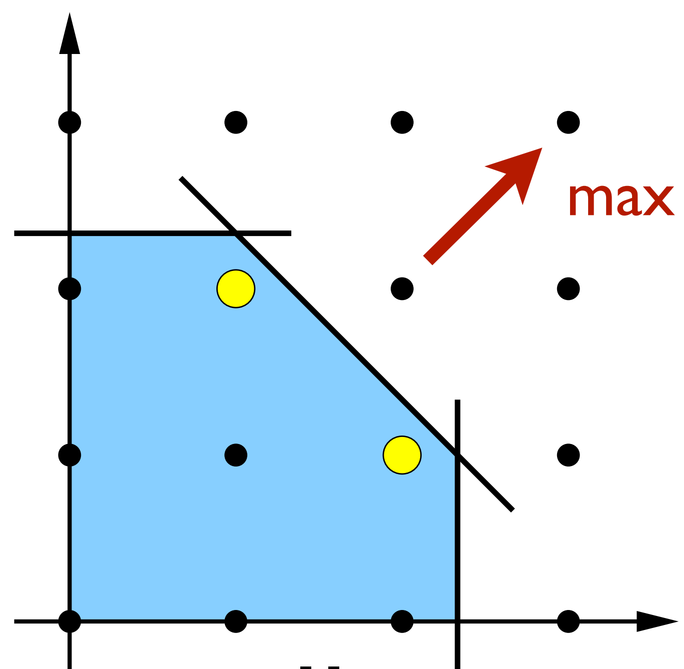
Polyhedral Problems



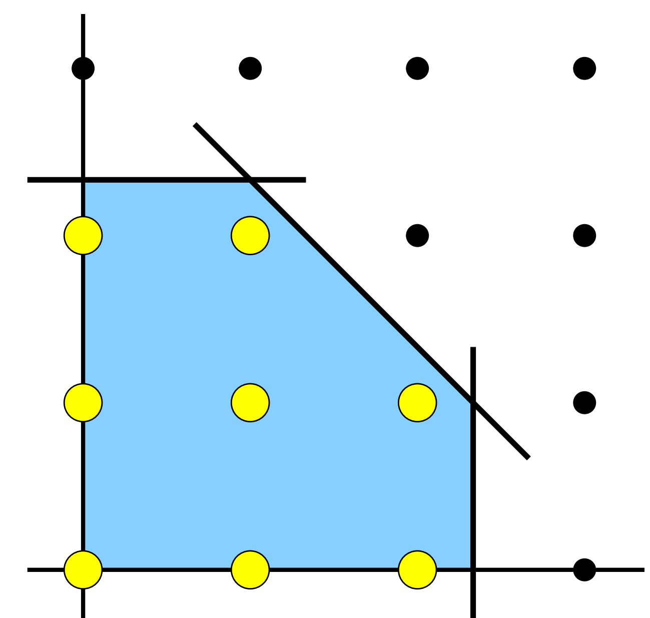
- I. Representation Conversion
- II. Integer Linear Programming
- III. Lattice Point Counting & Exact Volumes



I.



II.



III.

How to use symmetry ?

(DFG-Project SCHU 1503/6-1)

Why care?

Polyhedra in Optimization

in mixed integer linear programming (MILP)

- Used in Scheduling, Logistics, etc.
- Standard modeling often introduces symmetries
- Marc Pfetsch and Thomas Rehn (2016+):
  At least 209 of 353 MIPLIB 2010 instances have
  non-trivial permutation symmetries (up to group order 10^{68000})
- Bob Bixby (Aussois 2011, personal communication):

By exploiting symmetry, Gurobi currently has an
average performance improvement of 30% on its test instances.
However, the used methods are only very basic
and there is a lot of potential for future improvement.

CoFounder of
CPLEX and Gurobi

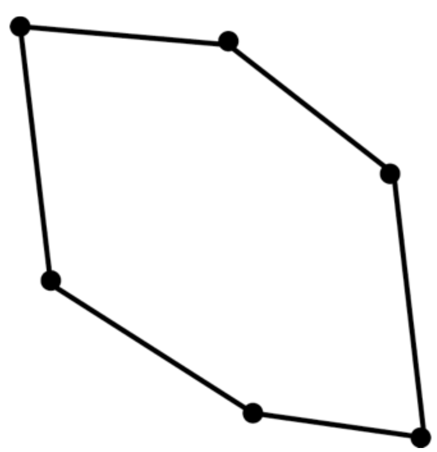
Prelude:

What are Polyhedral Symmetries?
...and how to compute them?

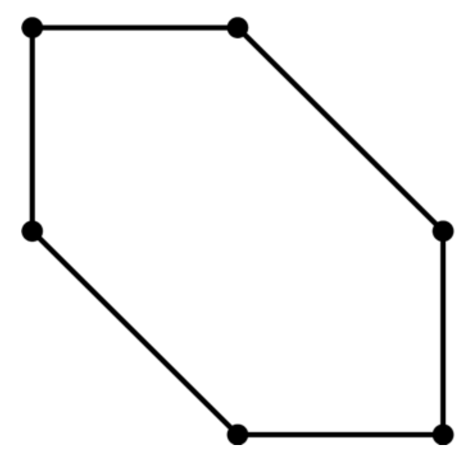
- David Bremner, Mathieu Dutour Sikiric, Dmitrii V. Pasechnik, Achill Schürmann, Thomas Rehn, Computing Symmetry Groups of Polyhedra, *LMS Journal of Computation and Mathematics*, 17 (2014), 565 - 581

Symmetry Groups

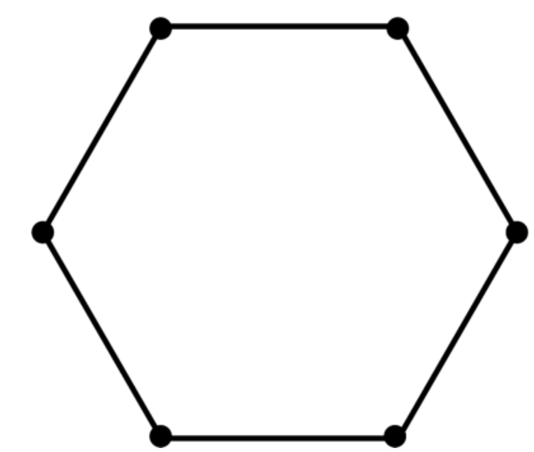
- Combinatorial, Linear, or Geometric Symmetries



$C_6 \rtimes C_2$
trivial
trivial



$C_6 \rtimes C_2$
$C_6 \rtimes C_2$
$C_2 \rtimes C_2$



$C_6 \rtimes C_2$
$C_6 \rtimes C_2$
$C_6 \rtimes C_2$

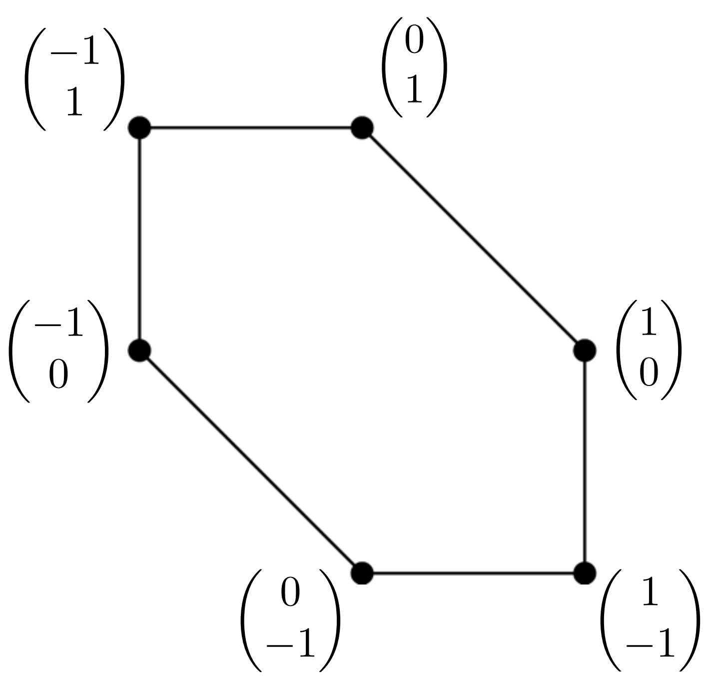
DEF: A linear automorphism of $\{v_1, \dots, v_m\} \subset \mathbb{R}^n$ is a regular matrix $A \in \mathbb{R}^{n \times n}$ with $Av_i = v_{\sigma(i)}$ for some $\sigma \in S_m$

Detecting Linear Automorphisms

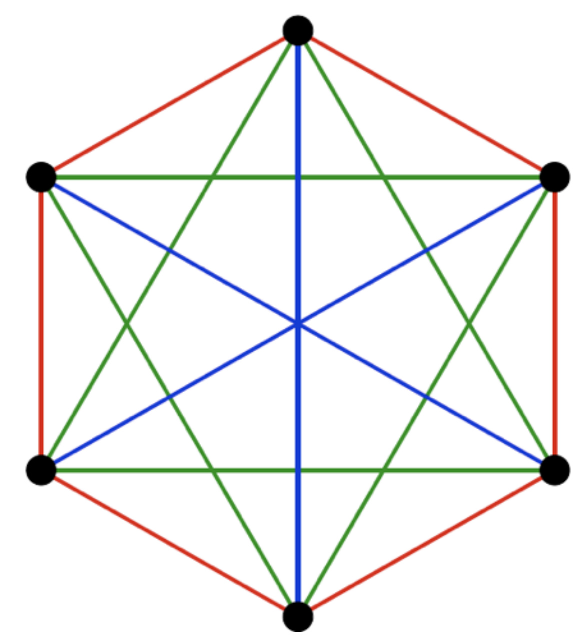
THM: The group of linear automorphisms is equal to

the automorphism group of the complete graph K_m

with edge labels $v_i^t Q^{-1} v_j$, where $Q = \sum_{i=1}^{m} v_i v_i^t$



$$Q = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$$



Sym Pol

uses PermLib or NAUTY by Brendan McKay
for computing automorphisms of colored graphs

A C++ Tool

also available through polymake

SymPol

- helps to compute linear automorphism groups
- converts representations using Recursive Decompositions

```
H-representation
begin
316 17 integer
0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
...
end
```

Getting the group:

```
sympol --automorphisms-only
```

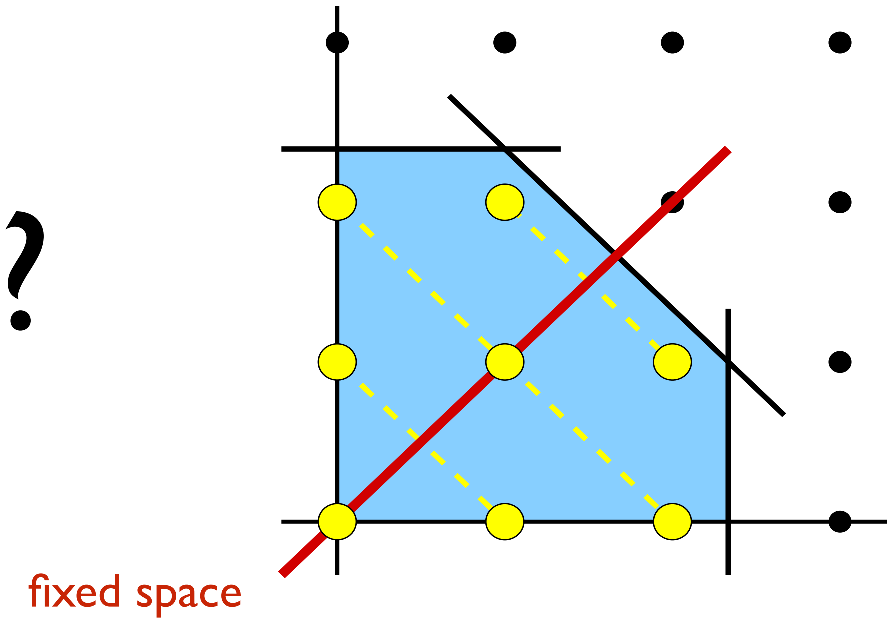
Getting vertices up to symmetry :

```
sympol --adm 40 input-file
```

```
permutation group
9
 3 5,7 9,11 14,13 16,19 21,23 25,27 30,29 32,
 ...
4
 33 17 49 308
```

```
V-representation
* UP TO SYMMETRY
begin
...
end
permutation group
* order 11520
* w.r.t. to the original inequalities/vertices
...
```

Detecting Linear Lattice Automorphisms?



PROB: We have no good general tools to compute linear lattice point preserving automorphisms of polytopes

or $\mathrm{GL}_n(\mathbb{Z})$-symmetries of a polytope P

$$\{M \in \mathrm{GL}_n(\mathbb{Z}) \ : \ MP = P\}$$

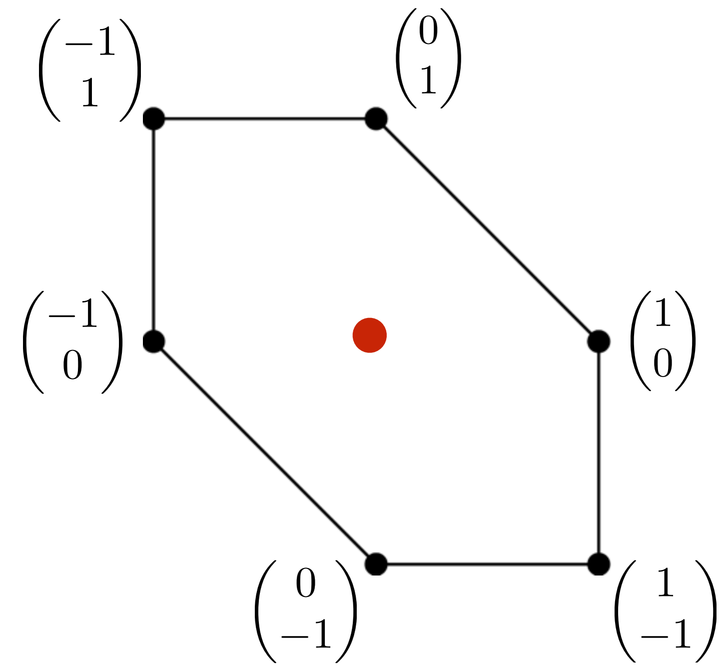
(coming with nice geometric properties)

EX: Among the 50 smallest MIPLIB instances (with $n \leq 1500$) six have $\mathrm{GL}_n(\mathbb{Z})$-symmetries that are no signed permutations!

Examples
(of Linear Lattice Automorphisms)

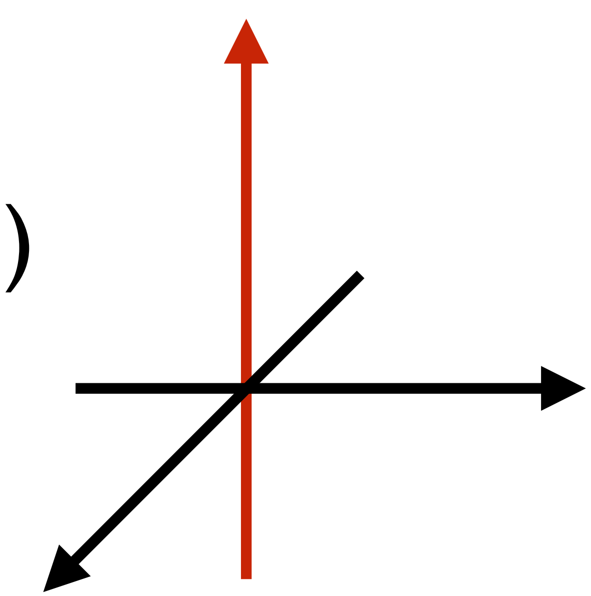
- $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in GL_2(\mathbb{Z})$ order 6

  fixed space $\{0\}$



- $\begin{pmatrix} 0 & \cdots & -1 & 0 \\ 1 & 0 & \cdots & 0 \\ & \ddots & & \\ 0 & \cdots & 1 & 1 \end{pmatrix} \in GL_n(\mathbb{Z})$ order 2(n-1)

  fixed space $\langle e_n \rangle$



Frontier I:

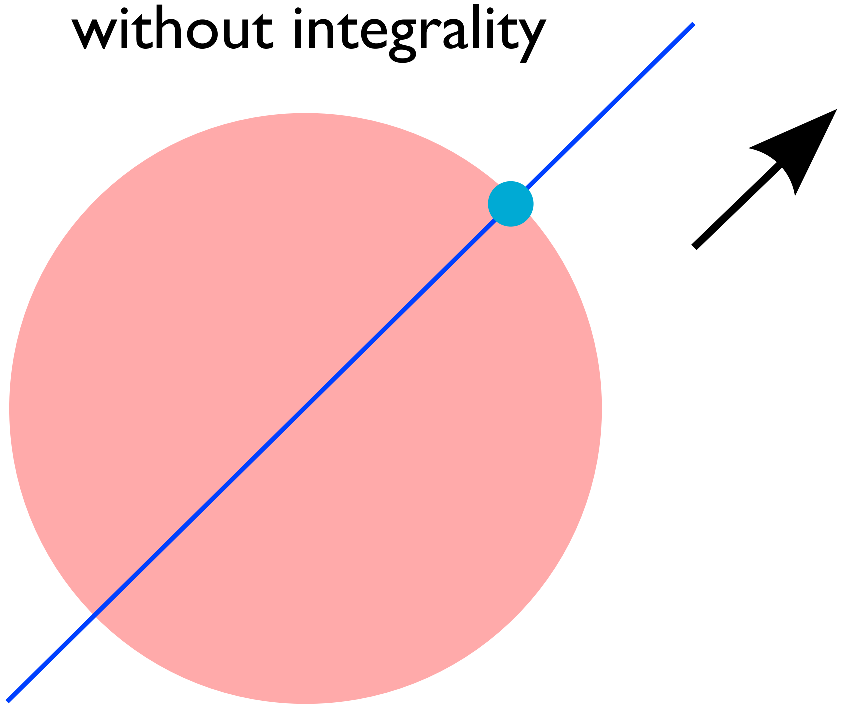
Exploiting Polyhedral Symmetries in Integer Convex Optimization

- Katrin Herr, Thomas Rehn and Achill Schürmann, Exploiting Symmetry in Integer Convex Optimization using Core Points, *Operations Research Letters*, 41 (2013), 298-304

- Katrin Herr, Thomas Rehn and Achill Schürmann, On Lattice-Free Orbit Polytopes, *Discrete & Computational Geometry*, 53 (2015), 144-172

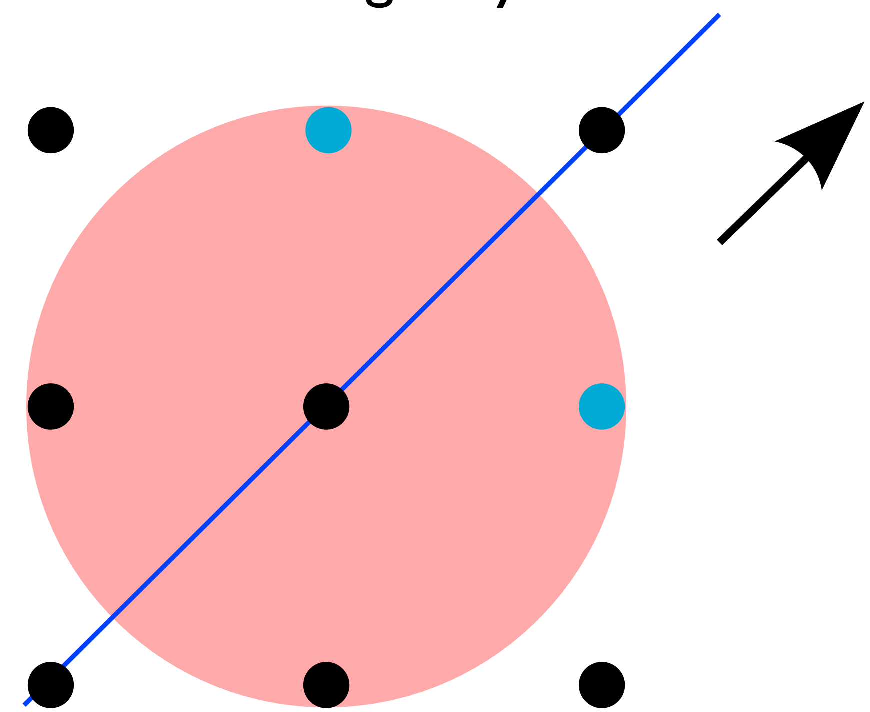
Convex Optimization

without integrality



Optimum attained within fixed subspace

with integrality



Optimum not necessarily attained in fixed subspace

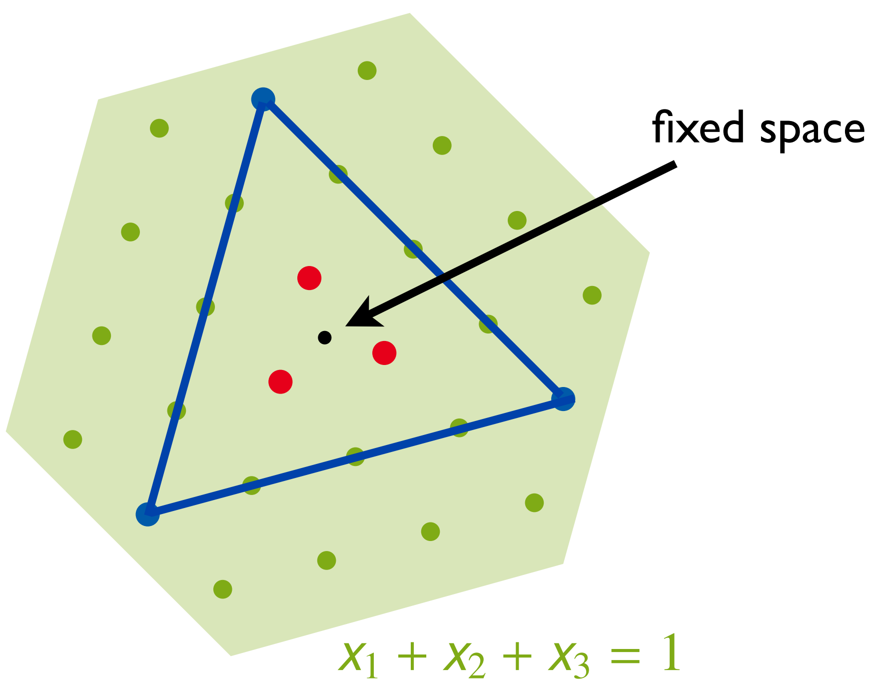
... with integrality constraints

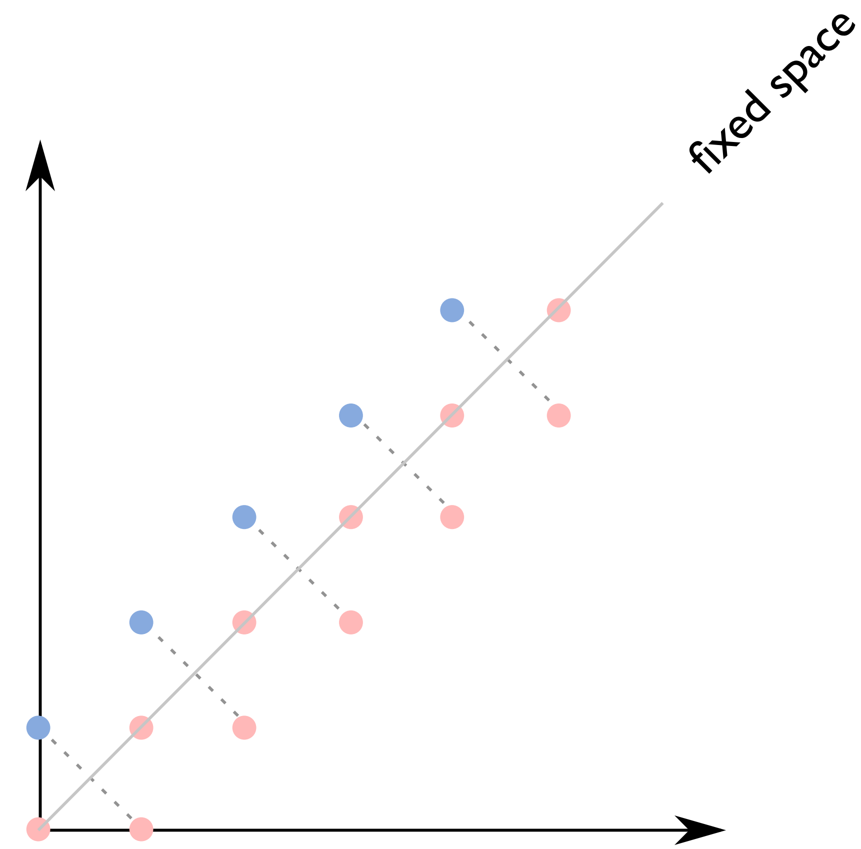
Core Points

(see Bödi, Herr, Joswig, *Math. Program. Ser. A*, 2013 for $\Gamma = S_n$)

DEF: $z \in \mathbb{Z}^n$ is a core point for $\Gamma \leq \mathrm{GL}_n(\mathbb{Z})$ if

$$\underline{(\mathrm{conv}\ \Gamma z)} \cap \mathbb{Z}^n = \Gamma z$$



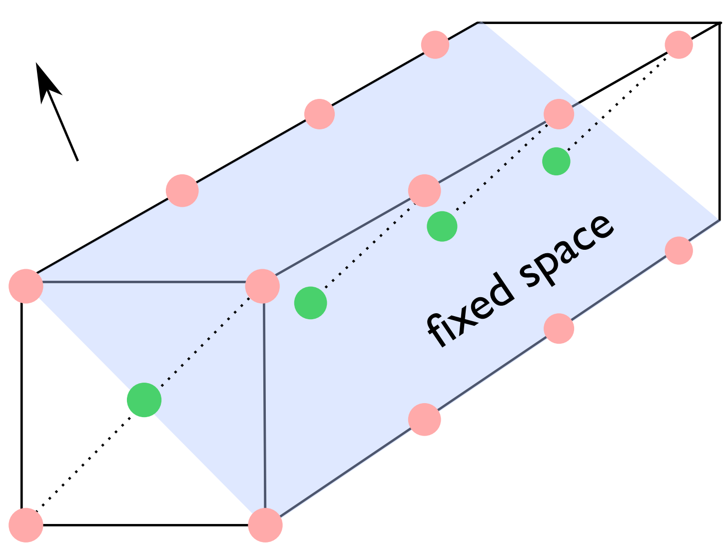


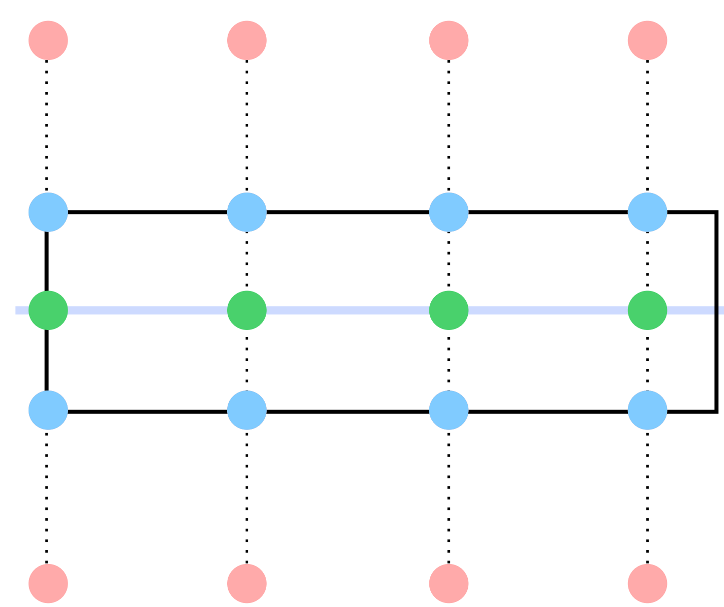
THM: If a Γ-invariant convex integer optimization problem has a solution, then a core point attains the optimal value.

(even a representative w.r.t. Γ)

Core Points of Symmetric Groups

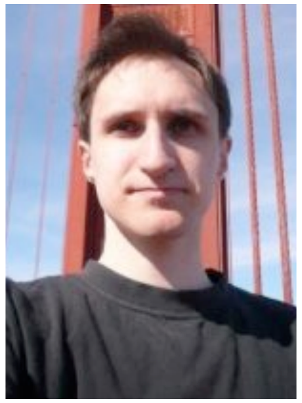
- For $\Gamma = S_n$ acting on coordinates of $\mathbb{R}^n$, all core points are 0/1-vectors up to translations by multiples of $\mathbb{1}$
- Core points of direct products are direct products of core points
- For $\Gamma = S_{n_1} \times \cdots \times S_{n_k}$ core points are 0/1-vectors up to translations of integral vectors from the fixed space
- Even naive enumeration approach beats commercial software





1. project polytope and $\mathbb{Z}^n$ onto fixed space
2. enumerate projected integer points in projected polytope
3. check feasibility of fibers by core sets

Rehn's reformulation idea



Thomas Rehn
(PhD 2014)

Core set-$\mathcal{V}$

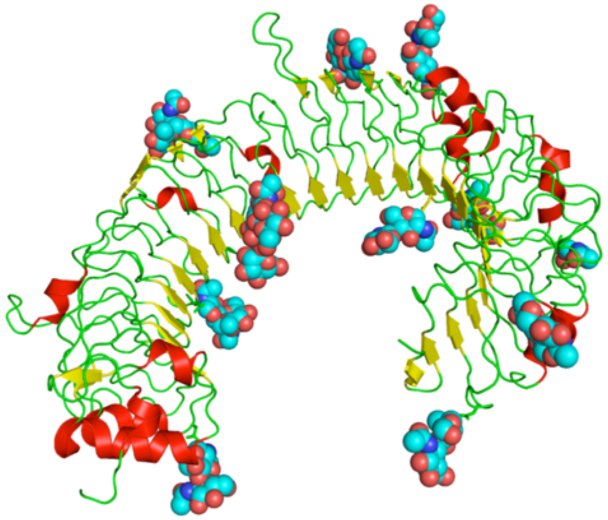
Let $c_1, \ldots, c_N$ be core set representatives. Then:

$$\mathrm{core}(\Gamma) \cong \left\{ \zeta_0 \mathbb{1} + \sum_{i=1}^{N} \zeta_i c_i \;:\; \zeta_0 \in \mathbb{Z},\ \zeta_i \in \{0,1\},\ \sum_{i=1}^{N} \zeta_i \leq 1 \right\}$$

- new IP-variables $\zeta_0, \zeta_1, \ldots, \zeta_N$
- for $\mathcal{S}_n$ or direct products thereof:
  same number of variables, $N = n - 1$

Solves "`toll-like`"

- open problem from MIPLIB 2010 collection
- 2883 binary variables, 4408 constraints
- automorphism group contains $(\mathcal{S}_2)^{230}$ as a subgroup
- after variable transformation and presolving there are 230 less variables and 460 less constraints
- transformed instance is solved by Gurobi 5.0 with 16 threads in about 18 hours



Toll-like receptor
(from Wikipedia)

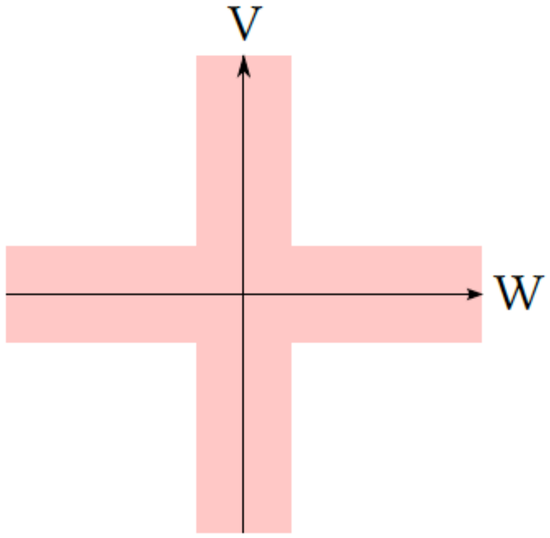
Transitive Permutation Groups

(with all coordinates in the same orbit)

- coming with a decomposition $\mathbb{R}^n = \bigoplus_{i=1}^{k} V_i$

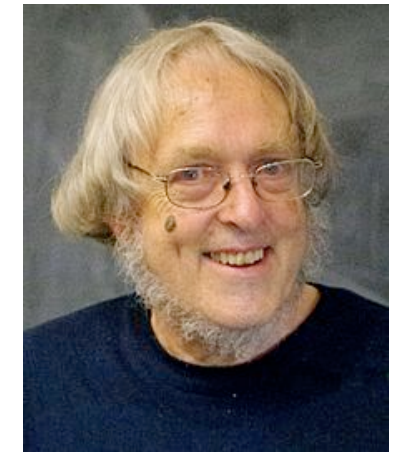
  with the V_i being Γ-invariant irreducible subspaces ($V_1 = \langle \mathbb{1} \rangle$)

THM: For $\Gamma \leq S_n$ acting transitive on coordinates of $\mathbb{R}^n$ there exists a constant $C(n)$, such that for every core point $z \in \mathbb{Z}^n$ there is a Γ-invariant subspace $V \neq \langle \mathbb{1} \rangle$ with $\|z|_V\| \leq C(n)$.



Finite vs. Infinite

(for transitive permutation groups)



COR: If Γ acts $\mathbb{R}$-irreducibly on $\mathbb{1}^{\perp}$, there exist only finitely many core points (up to translations by multiples of $\mathbb{1}$)

= 2-homogeneous

(Peter Cameron, 1972)

CONJECTURE:

All other transitive permutation groups have infinitely many core points up to translations by multiples of $\mathbb{1}$

- true for all groups with irrational invariant subspaces
- true for all imprimitive groups (with rational inv. subspaces)
- true for all primitive groups up to degree $n = 127$

Creating difficult IP-instances

using primitive permutation groups with infinite core sets

Table 7.2.: IP feasibility for orbit polytopes of primitive groups

		Gurobi		polymake & Gurobi		
Id	max $\lvert A_{ij}^{-} \rvert$	#nodes (10^6)	time (s)	#nodes (10^6)	time (s)	#subp.
15(5)	2851	252.0	6017.5	0.0	10.7	29
15(5)	11101	387.6	>10800.0	0.3	16.9	29
15(9)	2053	0.0	0.7	0.0	54.3	456
15(9)	7993	0.3	23.8	0.0	63.4	456
16(6)	2749	102.1	1905.2	0.0	6.4	24
16(6)	10681	548.7	>10800.0	0.0	6.5	24
16(9)	2713	0.4	21.9	0.0	38.2	280
16(9)	6013	3.3	96.9	0.0	39.3	280
21(8)	9352	35.7	1609.1	3.3	120.6	22
21(8)	36847	216.4	>10800.0	200.2	6765.7	22
21(8)	36847	216.4	>10800.0	69.6	1944.0	27
21(12)	287	1.0	57.1	0.2	34.8	150
21(12)	2155	242.9	>10800.0	74.8	3368.5	150
21(12)	2155	242.9	>10800.0	29.5	828.9	349

using Gurobi 5.5.0 on Intel Core-i7 with eight logical CPUs at 2.8GHz and 16 GB RAM

Frontier II:

Exploiting Polyhedral Symmetries in Lattice Point Counting and Computing Exact Volumes

- Achill Schürmann, Exploiting Polyhedral Symmetry in Social Choice, *Social Choice and Welfare,* 40 (2013), 1097-1110
- Erik Friese, William V. Gehrlein, Dominique Lepelley and Achill Schürmann, The impact of dependence among voters' preferences with partial indifference, *Quality & Quantity,* 2016+

Polyhedral Model in Social Choice

- Impartial Anonymous Culture (IAC) assumption:
  every voting situation is equally likely

- for three candidates a, b and c, let

n_{ab} number of voters with choice $\mathsf{a} > \mathsf{b} > \mathsf{c}$

n_{ac} number of voters with choice $\mathsf{a} > \mathsf{c} > \mathsf{b}$

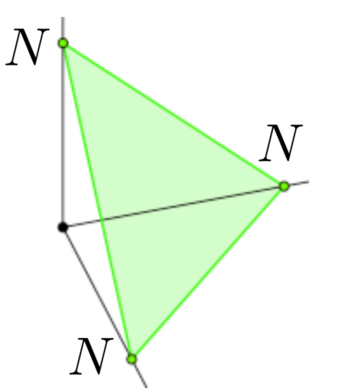
n_{ba} number of voters with choice $\mathsf{b} > \mathsf{a} > \mathsf{c}$

...

$(n_{\mathsf{ab}}, n_{\mathsf{ac}}, n_{\mathsf{ba}}, n_{\mathsf{bc}}, n_{\mathsf{ca}}, n_{\mathsf{cb}})$ describes a voting situation

$$N = n_{\mathsf{ab}} + n_{\mathsf{ac}} + n_{\mathsf{ba}} + n_{\mathsf{bc}} + n_{\mathsf{ca}} + n_{\mathsf{cb}}$$

is total number of voters



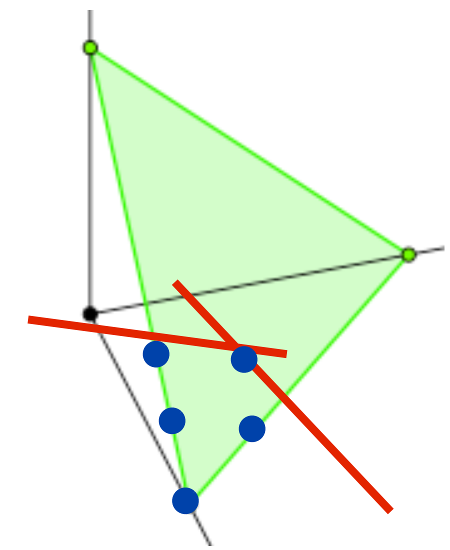
Counting Lattice Points

- Candidate a is a Condorcet winner if

(1) $n_{\text{ab}} + n_{\text{ac}} + n_{\text{ca}} \ > \ n_{\text{ba}} + n_{\text{bc}} + n_{\text{cb}}$ (a beats b)

(2) and $n_{\text{ab}} + n_{\text{ac}} + n_{\text{ba}} \ > \ n_{\text{ca}} + n_{\text{cb}} + n_{\text{bc}}$ (a beats c)

That is: $(n_{\text{ab}}, n_{\text{ac}}, n_{\text{ba}}, n_{\text{bc}}, n_{\text{ca}}, n_{\text{cb}}) \in \mathbb{Z}^6_{\geq 0}$



is in the polyhedron

$$P_N = \left\{ n \in \mathbb{R}^6 \ \middle| \ N = \sum_{\text{xy}} n_{xy}, \ n_{xy} \geq 0 \ \text{ and } \ (1), (2) \right\}$$

Likeliness of Condorcet paradox

Quasi-polynomial for $\#(P_N \cap \mathbb{Z}^6)$ can be obtained

using barvinok, Latte or Normaliz

```
  1/384 * N^5
+ ( -1/64 * {( 1/2 * N + 0 )} + 3/64 ) * N^4
+ ( -19/96 * {( 1/2 * N + 0 )} + 31/96 ) * N^3
+ ( -29/32 * {( 1/2 * N + 0 )} + 17/16 ) * N^2
+ ( -343/192 * {( 1/2 * N + 0 ) } + 5/3 ) * N
+ ( -83/64 * {( 1/2 * N + 0 )} + 1 )
```

(Number of voting situations with N voters and candidate a as Condorcet winner)

Likeliness of Condorcet Paradox $\quad 1 - 3\dfrac{\text{q-poly}}{\binom{N+5}{5}}$

For large elections $(N \to \infty)$:

$$1 - 3\frac{1/384}{1/120} = \frac{1}{16} = 0.0625$$

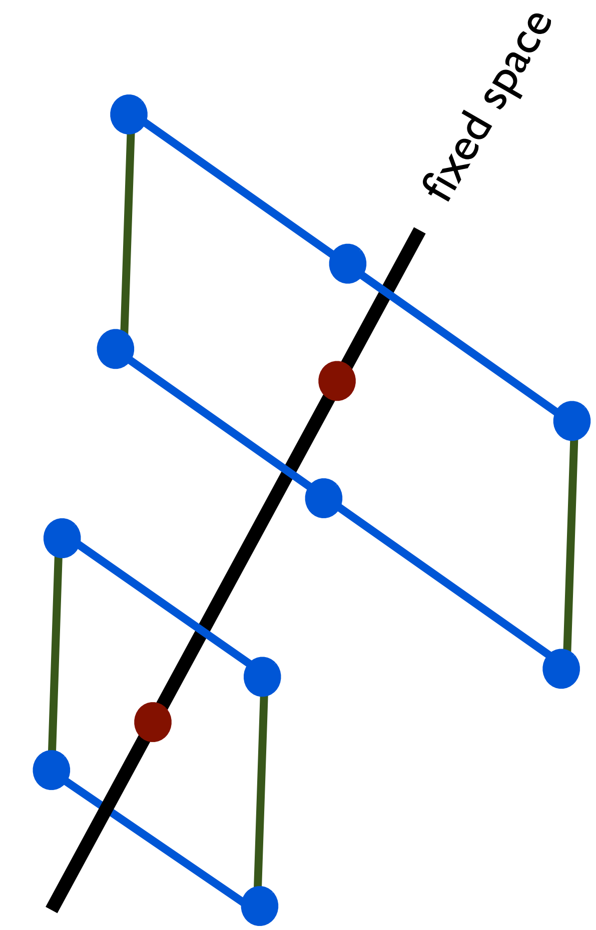
Grouping of variables

$$\boxed{n_{\mathsf{a}}} + n_{\mathsf{ca}} > n_{\mathsf{ba}} + \boxed{n_{\mathsf{R}}}$$

$$\boxed{n_{\mathsf{a}}} + n_{\mathsf{ba}} > n_{\mathsf{ca}} + \boxed{n_{\mathsf{R}}}$$

$$N = \boxed{n_{\mathsf{a}}} + n_{\mathsf{ba}} + n_{\mathsf{ca}} + \boxed{n_{\mathsf{R}}}$$

$$\boxed{n_{\mathsf{a}}} \qquad \boxed{n_{\mathsf{R}}}$$



$(n_{\mathsf{a}}, n_{\mathsf{ba}}, n_{\mathsf{ca}}, n_{\mathsf{R}})$ describes $(n_{\mathsf{a}}+1)(n_{\mathsf{R}}+1)$ voting situations

(former lattice points)

THUS: the polytope decomposes into fibers of simplotopes (cross products of simplices)

The next generation Ehrhart theory
Counting with polynomial weights

- Two methods:
  - via rational generating functions
  - via local Euler-Maclaurin formula

Baldoni, Berline, Vergne, 2009

- "experimental" implementation
  available in barvinok

- since May 2013 in Normaliz and
  since Aug 2013 in LattE integrale

Verdoolaege

Bruns

Köppe

DeLoera

Using local formulas

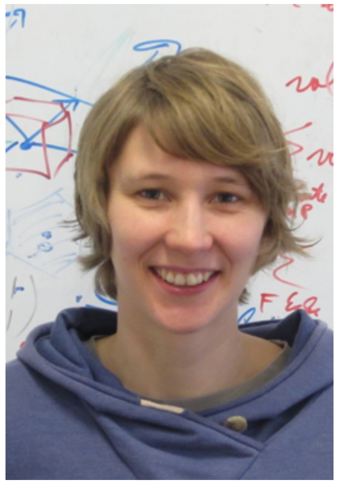
$$\#\left(P \cap \mathbb{Z}^n\right) \;=\; \sum_{F \text{ face of } P} \theta(P,F) \cdot \mathrm{relvol}(F)$$

with $\theta(P,F)$ depending only on the outer normal cone of P at F

(Morelli, McMullen, 1993)

There are many different choices for θ:

- Pommersheim and Thomas, 2004
- $O_n(\mathbb{Z})$ invariant, Berline and Vergne, 2007
- invariant with respect to a given group $\Gamma \leq \mathrm{GL}_n(\mathbb{Z})$



Maren

Conclusions?

… a lot TODOs

- ANALYZE GROUPS
  compute and analyze more (mixed) integer linear symmetry groups of symmetric lattice polytope problems

- EXTEND THEORY
  classify / approximate core points for interesting groups; obtain symmetric decompositions and invariant local formulas

- NEW ALGORITHMS
  create new algorithms and heuristics that exploit knowledge about core points, respectively symmetric decompositions