

Combinatorial mixed valuations on polytopes

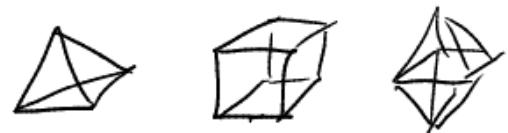


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joint with K. Jachymski (TU Vienna)

Polytopes and Valuations

$$\Lambda = \mathbb{R}^d, \mathbb{Z}^d$$

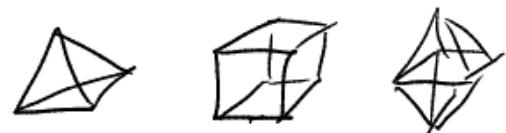


$$\mathcal{P}(\Lambda) := \{ P \subset \mathbb{R}^d : P \text{ polytope w/ vertices in } \Lambda \}$$

$\mathcal{P}(\mathbb{R}^d)$ - general polytopes , $\mathcal{P}(\mathbb{Z}^d)$ - lattice polytopes

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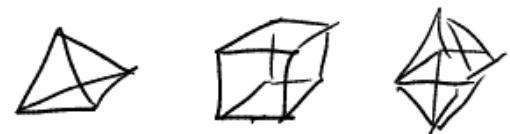
G abelian group , $\varphi : \mathcal{P}(\Lambda) \rightarrow G$ Λ -valuation if

$$\varphi(P \cup Q) = \varphi(P) + \varphi(Q) - \varphi(P \cap Q)$$

whenever $P, Q, P \cup Q, P \cap Q \in \mathcal{P}(\Lambda)$

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whenever $P, Q, P \cup Q, P \cap Q \in \mathcal{P}(\Lambda)$ and

$$\varphi(\tau + P) = \varphi(P) \quad P \in \mathcal{P}(\Lambda), \tau \in \Lambda$$

\mathbb{R}^d -valuations: volume, surface area, Euler characteristic, mean width...

\mathbb{Z}^d -valuations: discrete volume $E(P) = |P \cap \mathbb{Z}^d|$

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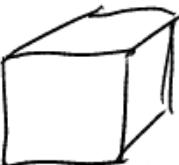
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Geometry

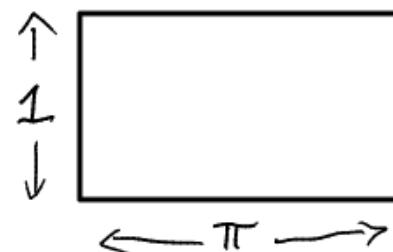
Valuations

Combinatorics

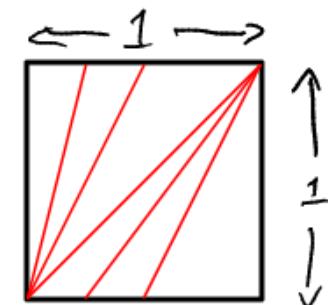
Hilbert's 3rd



equidissectable?



dissectable into
squares?



dissectable into odd #
of equal-area triangles?

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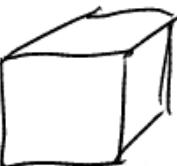
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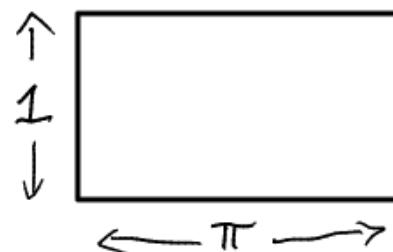
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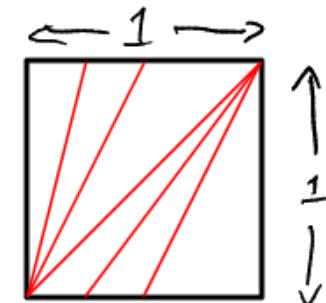


equidissectable?



dissectable into squares?

No!



dissectable into odd # of equal-area triangles?

No!

→ Hadwiger: P, Q equidissectable

$\Leftrightarrow \varphi(P) = \varphi(Q)$ for all rigid-motion invariant simple valuations

Volumes & mixed volumes

$P_1, \dots, P_d \subseteq \mathbb{R}^d$ polytopes

$$\text{vol}(\lambda_1 P_1 + \dots + \lambda_d P_d) = \sum_{j_1, \dots, j_d=1}^d MV(P_{j_1}, \dots, P_{j_d}) \lambda_{j_1} \dots \lambda_{j_d}$$

Mixed volumes

algebraically: $MV(P_1, P_2, \dots, P_d)$ is the polarization of $\text{vol}(\cdot)$

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deep theory of mixed volumes: Brunn-Minkowski, Alexander-Fenchel,

BKK Theorem: $MV(P_1, \dots, P_d) = \#\{x \in (\mathbb{C}^*)^d : f_1(x) = \dots = f_d(x) = 0\}$

$$f_i \in \mathbb{C}[x_1, \dots, x_n]$$

$P_i = \text{Nev}_i(f_i)$

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$P_i = \text{Nest}(f_i)$

Fundamental: nonnegativity / monotonicity

$$0 \leq MV(P_1, P_2, \dots, P_d) \leq MV(Q_1, Q_2, \dots, Q_d)$$

$$\text{for } P_1 \subseteq Q_1, \dots, P_d \subseteq Q_d$$

$P_1, P_2, \dots, P_r \subset \mathbb{R}^d$ lattice polytopes (Bernstein'76, McMullen'77)

$$E(uP_1 + \dots + u_r P_r) = |\mathbb{Z}^d \cap (u_1 P_1 + \dots + u_r P_r)| \text{ polynomial, } \deg \leq d$$

Ehrhart polynomial

$$E(nP) = e_r(P) n^r + e_{r-1}(P) n^{r-1} + \dots + e_0(P)$$

$r = \dim P$

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Theory of mixed discrete volumes?

Polarization $\leadsto ME(P_1, \dots, P_k)$ symmetric + multi-linear

$e_r = \text{vol}$, $e_0 = \text{Euler char}$, e_i homogeneous deg = i

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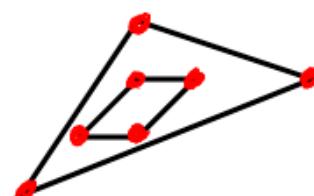
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» e_i : NOT monotone: Pick's formulae



» e_i : NOT nonnegative: Reeve tetrahedra

E is NOT homogeneous!

Combinatorial mixed valuations

$$CM_k \varphi(P_1, P_2, \dots, P_r) := \sum_{I \subseteq \{1, \dots, r\}} (-1)^{r - |I|} \varphi\left(\sum_{i \in I} P_i\right)$$

$$CM_1 \varphi(P) = \varphi(P) - \varphi(\{\emptyset\}) , \quad CM_2 \varphi(P, Q) = \varphi(P+Q) - \varphi(P) - \varphi(Q) + \varphi(\{\emptyset\})$$

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Coefficients of φ in binomial basis

$$\varphi(u_1 P_1 + \dots + u_r P_r) = \sum_{\alpha_1, \dots, \alpha_r \geq 0} CM \varphi\left(\underbrace{P_1, \dots, P_1}_{\alpha_1}, \dots, \underbrace{P_r, \dots, P_r}_{\alpha_r}\right) \binom{u_1}{\alpha_1} \binom{u_2}{\alpha_2} \dots \binom{u_r}{\alpha_r}$$

$$P_1, \dots, P_r \in \mathbb{R}^d, r > d \text{ then } CM_r \varphi(P_1, \dots, P_r) \equiv 0.$$

→ algebraic characterization of $CM \varphi$

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Better reflect properties of φ in inhomogeneous case !

Minkowski: $P_1, P_2, \dots, P_d \subset \mathbb{R}^d$ polytopes

$$CMV(P_1, \dots, P_d) = \sum_{\mathcal{I}} (-1)^{d - |\mathcal{I}|} \text{vol}\left(\sum_{i \in \mathcal{I}} P_i^\circ\right) = d! MV(P_1, \dots, P_d)$$

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Bernstein'76: P_1, \dots, P_d lattice polytopes

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$$E\left(\begin{array}{c} \text{blue shaded pentagon} \\ \text{red shaded triangle} \end{array}\right) - E\left(\begin{array}{c} \text{blue shaded triangle} \\ \text{red shaded triangle} \end{array}\right) - E\left(\begin{array}{c} \text{red shaded triangle} \\ \bullet \end{array}\right) + E(\bullet) = 2$$

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$$E(\text{Diagram}) - E(\text{Triangle}) - E(\text{V-shape}) + E(\bullet) = 2$$

Bihan '16: $CME(P_1, \dots, P_r) \geq 0$ for any P_1, \dots, P_r lattice polytopes

CME monotone?

For general 1-valuations?

Interlude: Combinatorial positive valuations

Ehrhart h^* -vector: $P \subset \mathbb{R}^d$ r -dim lattice polytope

$$\sum_{n \geq 0} E(nP) z^n = \frac{h_0^* + h_1^* z + \dots + h_r^* z^r}{(1-z)^{r+1}}$$

$\leftarrow h^*$ -polynomial/
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h^φ -vector

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Thm. (JS'15)

$$h^\varphi(P) \geq 0 \iff h^\varphi(P) \leq h^\varphi(Q) \iff \varphi(\text{relint } \Delta) \geq 0$$

$$P \subseteq Q$$

all lattice simplices Δ

Theorem. (JS'16) If comb. pos Λ -valuation. Then

$$0 \leq CM\varphi(P_1, \dots, P_r) \leq CM\varphi(Q_1, \dots, Q_r)$$

CM-monotone

for all $P_i \subseteq Q_1, \dots, P_r \subseteq Q_r$.

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Cor. If φ is nonnegative and simple ($\varphi(P)=0$ if P not full-dim)
then φ is CM-monotone.

Cor. Combinatorial mixed volumes $CMVol$ are monotone + nonnegative.

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Cor. The discrete volume E is CM-monotone.

Cor(Bihan'16). $CME(P_1, \dots, P_r) = \sum_I (-1)^{r-|I|} |\mathbb{Z}^d \cap \sum_{i \in I} P_i| \geq 0$.

Monotonicity and CM-monotonicity

φ CM-monotone $\implies \varphi$ monotone

$$CM, \varphi(P) = \varphi(P) - \varphi(\{0\}) \leq \varphi(Q) - \varphi(\{0\}) = CM, \varphi(Q)$$

for $P \subseteq Q$

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Combinatorial counterpart to result of Berzig - Fu'11:

Thm. φ \mathbb{R}^d -valuation. $\varphi = \varphi_0 + \dots + \varphi_d$, φ_i : homogeneous

φ monotone \iff $\varphi_0, \dots, \varphi_d$ monotone.

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$$f \text{ CM-monotone} \implies f \text{ monotone}$$

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► **No** invariance under rigid motions required! (with rigid motion invariance **easy!**)

► Results are equivalent: JS \iff Beruig - Fu

Conjecture. For $\Lambda = \mathbb{Z}^d$. For φ \mathbb{Z}^d -valuation

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Focus on *lattice-invariant* ($SL_d(\mathbb{Z})$) \mathbb{Z}^d -valuations

Thm (Betke-Kneser '85). $CME_0(P), CME_1(P), \dots, CME_d(P)$ form
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Open: Conjecture holds for lattice inv \mathbb{Z}^3 -valuations ?

Computational aspects

Prop. $P_1, \dots, P_d \subset \mathbb{R}^d$ polytopes. $MV(P_1, \dots, P_d) > 0 \iff$

Exist linear independent line segments $S_1 \subset P_1, \dots, S_d \subset P_d$

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Then Dyer-Gritzmann-Hufnagel holds for $CM\varphi(P_1, \dots, P_r) = 0$.

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Prop. $P_1, \dots, P_r \in \mathcal{P}(\mathbb{Z}^d)$, $S_1 \subset P_1, \dots, S_r \subset P_r$ lin indep lattice simplices

$$CME(P_1, \dots, P_r) \geq \dim S_1 \cdot \dim S_2 \cdot \dots \cdot \dim S_r$$

Interlude: Motivic arithmetic genus*

$f_1, f_2, \dots, f_r \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ generic Laurent polynomials
with Newton-Polytopes $P_1, \dots, P_r \subset \mathbb{R}^n$

$Y := \{x \in (\mathbb{C}^*)^n : f_1(x) = \dots = f_r(x) = 0\}$ non-compact complete intersection

$e^{0,+}(Y) =$ motivic arithmetic genus of Y

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* go on, ask Christian or Berjouini

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Thm (Di Rocco, Haase, Nill '16).

$$e^{o,+}(Y) = \sum_{I \subseteq [r]} (-1)^{n - |I|} |\mathbb{Z}^n \cap \sum_I P_i| = (-1)^{n-r} CME(P_1, \dots, P_r)$$

* go on, ask Christian or Berjouini

Interlude: Motivic arithmetic genus*

$f_1, f_2, \dots, f_r \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ generic Laurent polynomials
with Newton-Polytopes $P_1, \dots, P_r \subset \mathbb{R}^n$

$Y := \{x \in (\mathbb{C}^*)^n : f_1(x) = \dots = f_r(x) = 0\}$ non-compact complete intersection

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Recall $E(u_1 P_1 + \dots + u_r P_r) = \sum_{\alpha} CME(\underbrace{P_1}_{\alpha_1}, \dots, \underbrace{P_r}_{\alpha_r}) \binom{u_1}{\alpha_1} \dots \binom{u_r}{\alpha_r}$

Cor (DHN'16). $e^{o,+}(Y) \geq 0$ and > 0 if $Y \neq \emptyset$

* go on, ask Christian or Berjouini

Cones in the polytope algebra

$\mathbb{Z}\mathcal{P}(\underline{A})$ free abelian group w/ generators $e_P \quad P \in \mathcal{P}(\underline{A})$

$$U = \mathbb{Z} \left\{ e_{P \cup Q} - e_P - e_Q + e_{P \cap Q}, \quad e_{P+t} - e_P \right\}_{\substack{P \cup Q, P, Q, P \cap Q \in \mathcal{P}(\underline{A}), \\ t \in \underline{A}}}, \quad P \in \mathcal{P}(\underline{A}), \quad t \in \underline{A}$$
$$\subseteq \mathbb{Z}\mathcal{P}(\underline{A})$$

Polytope algebra $T^* := \mathbb{Z}\mathcal{P}(\underline{A})/U$

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Polytope algebra $T^* := \mathbb{Z}\mathcal{P}(\mathbb{A})/U$

$$\boxed{\dots} = \boxed{} - \boxed{} - \boxed{} + \boxed{}$$

Cones in the polytope algebra

$\mathbb{Z}\mathcal{P}(\Lambda)$ free abelian group w/ generators $e_P \quad P \in \mathcal{P}(\Lambda)$

$$U = \mathbb{Z} \left\{ e_{P \cup Q} - e_P - e_Q + e_{P \cap Q}, \quad e_{P+t} - e_P \right. \\ \left. P \cup Q, P, Q, P \cap Q \in \mathcal{P}(\Lambda), \quad P \in \mathcal{P}(\Lambda), t \in \Lambda \right\} \subseteq \mathbb{Z}\mathcal{P}(\Lambda)$$

Polytope algebra $\mathbb{T} := \mathbb{Z}\mathcal{P}(\Lambda)/U$

$$\square = \boxed{\square} - \boxed{} - \boxed{} + \boxed{\bullet}$$

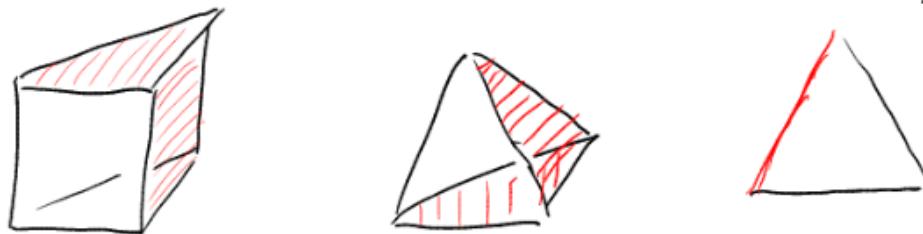
Universal Λ -valuation $P \mapsto [\![P]\!] + U \in \mathbb{T}$

$$\begin{array}{ccc} \mathcal{P}(\Lambda) & \xrightarrow{\iota} & G \\ \downarrow & & \dashrightarrow \\ \mathbb{T} & \dashrightarrow & \bar{\mathcal{P}} \end{array}$$

Comb. positive, monotone- and CM-monotone valuations are
duals/polars to cones in Π .

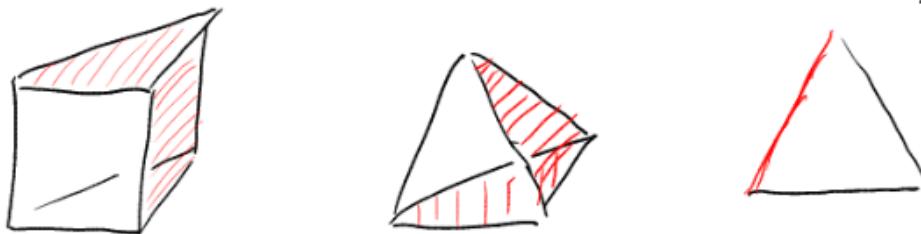
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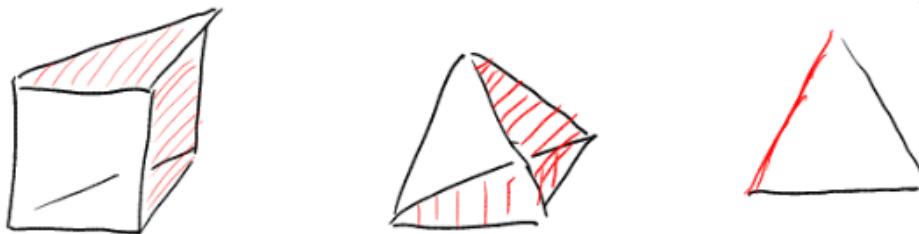
Thm. (JS'16). $P_i, Q_i \in \mathcal{P}(A)$, $P_i \subseteq Q_i$, $i = 1, 2, \dots, r$

$$[n_1 Q_1 + n_2 Q_2 + \dots + n_r Q_r] - [n_1 P_1 + \dots + n_r P_r] = \sum_{\alpha} \zeta_{\alpha} \binom{n_1}{\alpha_1} \binom{n_2}{\alpha_2} \dots \binom{n_r}{\alpha_r}$$

with $\zeta_{\alpha} \in \mathcal{L}$

Comb. positive, monotone- and CM-monotone valuations are duals/polars to cones in \mathbb{T} .

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Thm. (JS'16). $P_i, Q_i \in \mathcal{P}(\Delta)$, $P_i \subseteq Q_i$, $i = 1, 2, \dots, r$

$$\llbracket n_1 Q_1 + n_2 Q_2 + \dots + n_r Q_r \rrbracket - \llbracket n_1 P_1 + \dots + n_r P_r \rrbracket = \sum_{\alpha} \sum_{\Sigma_\alpha} \binom{n_1}{\alpha_1} \binom{n_2}{\alpha_2} \dots \binom{n_r}{\alpha_r}$$

with $\Sigma_\alpha \in \mathcal{L}$



φ comb pos ($\varphi(\text{relint } \Delta) \geq 0$) $\Rightarrow \varphi(\Sigma) \geq 0$ for all $\Sigma \in \mathcal{L}$.

