Dilated Floor Functions and Their Commutators

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Topics Covered

- Part I. Dilated floor functions
- Part II. Dilated floor functions that commute
- Part III. Dilated floor functions with positive commutators
- Part IV. Concluding remarks

- Takumi Murayama, Jeffrey C. Lagarias and D. Harry Richman,
Dilated Floor Functions that Commute,
American Math. Monthly **163** (2016), No. 10, to appear.
(`arXiv:1611.05513, v1`)

- J. C. Lagarias and D. Harry Richman,
Dilated Floor Functions with Nonnegative Commutators, preprint.

- J. C. Lagarias and D. Harry Richman,
Dilated Floor Functions with Nonnegative Commutators II: Third Quadrant Case, in preparation.

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Part 1. Dilated Floor Functions

- We start with the floor function $\lfloor x \rfloor$.

- The *floor function* discretizes the real line, rounding a real number x down to the nearest integer:

$$x = \lfloor x \rfloor + \{x\}$$

  where $\{x\}$ is the fractional part function, i.e. x (modulo one).

- The *ceiling function* which rounds up to the nearest integer is conjugate to the floor function:

$$\lceil x \rceil = -\lfloor -x \rfloor \qquad [= R \circ \lfloor \cdot \rfloor \circ R\,(x)\ ]$$

  using the conjugacy function $R(x) = -x$.

Dilated Floor Functions-1

- The dilations for $\alpha \in \mathbb{R}^*$ act on the real line as

  $$D_\alpha(x) = \alpha x,$$

  They act under composition as the multiplicative group $GL(1, \mathbb{R}) = \mathbb{R}^*$.

- A *dilated floor function* with real *dilation factor* α is defined by

  $$f_\alpha(x) := \lfloor \alpha x \rfloor \qquad [= \lfloor \cdot \rfloor \circ D_\alpha(x)]$$

- We allow negative α, so we are able to build ceiling functions.

Dilated Floor Functions-2

• Dilated floor functions encode information on the Riemann zeta function.

• Its Mellin transform is given when $\alpha > 0$, for $Re(s) > 1$, by

$$\int_0^\infty \lfloor \alpha x \rfloor x^{-s-1} dx = \alpha^s \frac{\zeta(s)}{s}$$

Also for $0 < Re(s) < 1$,

$$\int_0^\infty \{\alpha x\} x^{-s-1} dx = -\alpha^s \frac{\zeta(s)}{s}$$

These two integrals are related via a (renormalized) integral which converges nowhere: $\int_0^\infty \alpha x \cdot x^{s-1} dx := 0$.

• Dilations are compatible with the Mellin transform. The Mellin transform preserves the $GL(1)$ scaling since $\frac{dx}{x}$ is unchanged by dilations.

Dilated Floor Functions -3

- Dilated floor functions can be used in describing data about lattice points in lattice polytopes, in the recently introduced notion of **intermediate Ehrhart quasi-polynomials** of the polytope. (Work of Baldoni, Berline, Köppe and Vergne, Mathematika **59** (2013), 1–22, and sequel with de Loera, Mathematika **62** (2016), 653–684).

- Their definition 23: For integer $q \geq 1$, let $\lfloor n \rfloor_q := q \lfloor \frac{1}{q} n \rfloor$ and let $\{n\}_q := n \pmod{q}$ (least nonnegative residue), so that

$$n = \lfloor n \rfloor_q + \{n\}_q.$$

- Their Table 2 gives examples of representing intemediate Ehrhart quasipolynomials in terms of such functions, in which the dilated functions $(\{4t\}_1)^2$ and $\{-5t\}_2$ appear.

Dilated Floor Functions -4

- Dilated functions without the discretization: *linear functions*

$$\ell_\alpha(x) = \alpha x.$$

- **Fact.** *Linear functions commute under composition, and satisfy for all* $\alpha, \beta \in \mathbb{R}$*,*

$$\ell_\alpha \circ \ell_\beta(x) = \ell_\beta \circ \ell_\alpha(x) = \ell_{\alpha\beta}(x).$$

for all $x \in \mathbb{R}$*.*

- **General Question.** Discretization destroys the convexity of linear functions. It generally destroys commutativity under composition. What properties remain?

Part II. Dilated Floor Functions that Commute

- **Question:** *When do dilated floor functions commute under composition of functions?*

- The question turns out to have an interesting answer.

Main Theorem

• **Theorem.** (L-M-R (2016)) (Commuting Dilated Floor Functions)
The set of (α, β) for which the dilated floor functions commute

$$\lfloor \alpha \lfloor \beta x \rfloor \rfloor = \lfloor \beta \lfloor \alpha x \rfloor \rfloor \quad \textit{for all} \quad x \in \mathbb{R},$$

consists of:

(i) Three one-parameter continuous families: $\alpha = 0$ or $\beta = 0$, or $\alpha = \beta$.

(ii) A two-parameter discrete family: $\alpha = \frac{1}{m}$ and $\beta = \frac{1}{n}$ for all integers $m, n \geq 1$.

Remark. In case (ii), setting $\mathbf{T}_m(x) := \lfloor \frac{1}{m} x \rfloor$ we have

$$\mathbf{T}_m \circ \mathbf{T}_n(x) = \mathbf{T}_n \circ \mathbf{T}_m(x) = \mathbf{T}_{mn}(x) \quad \text{for all} \quad x \in \mathbb{R}.$$

for all $m, n \geq 1$. (These are same relations as for linear functions.)

The Discrete Commuting Family

• **Claim:** *Suppose* $m, n \geq 1$ *are integers. Then* $\lfloor \frac{1}{m} \lfloor \frac{1}{n} x \rfloor \rfloor = \lfloor \frac{1}{mn} x \rfloor$.

• Exchanging m and n, the claim implies $\lfloor \frac{1}{m} \lfloor \frac{1}{n} x \rfloor \rfloor = \lfloor \frac{1}{n} \lfloor \frac{1}{m} x \rfloor \rfloor$, which gives the commuting family.

• To prove the claim: The functions are step functions and agree at $x = 0$. We study where and how much the functions jump. The right side $\lfloor \frac{1}{mn} x \rfloor$ jumps exactly at x an integer multiple of mn, and the jump is of size 1 .

• For the left side $\lfloor \frac{1}{m} \lfloor \frac{1}{n} x \rfloor \rfloor$, the inner function $\lfloor \frac{1}{n} x \rfloor$ is always an integer, and it jumps by 1 at integer multiples of n. Now the outer function jumps exactly when the k-th integer multiple of n (of the inner function) has k divisible by m. So it jumps exactly at multiples of mn and the jump is of size 1. QED.

Proof Method: Analyze Upper Level Sets

- **Definition.** The *upper level set* $S_f(y)$ of a function $f : \mathbb{R} \to \mathbb{R}$ is

$$S_f(y) := \{x : f(x) \geq y\}.$$

- It is a closed set for the floor function (but not for the ceiling function).

Example. For the composition of dilated floor functions $f_\alpha \circ f_\beta(x) = \lfloor \alpha \lfloor \beta x \rfloor \rfloor$ we use notation: $S_{\alpha,\beta}(y) := \{x : \lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq y\}$.

- **Key Lemma.** *For $\alpha > 0, \beta > 0$ and n an integer, the upper level set at level $y = n$ is the closed set*

$$S_{\alpha,\beta}(n) = [\, \frac{1}{\beta} \lceil \frac{1}{\alpha} n \rceil, +\infty).$$

Upper Level Sets-2

Key Equivalence: For y equal to an *integer* n the upper level set is

$$
\begin{aligned}
x \in S_{\alpha,\beta}(n) &\Leftrightarrow \lfloor \alpha \lfloor \beta(x) \rfloor \rfloor \geq n \quad \text{(the definition)} \\
&\Leftrightarrow \alpha \lfloor \beta x \rfloor \geq n \quad \text{(the right side is in } \mathbb{Z}\text{)} \\
&\Leftrightarrow \lfloor \beta x \rfloor \geq \frac{1}{\alpha} n \quad \text{(since } \alpha > 0\text{)} \\
&\Leftrightarrow \lfloor \beta x \rfloor \geq \lceil \frac{1}{\alpha} n \rceil \quad \text{(the left side is in } \mathbb{Z}\text{)} \\
&\Leftrightarrow \beta x \geq \lceil \frac{1}{\alpha} n \rceil \quad \text{(the right side is in } \mathbb{Z}\text{)} \\
&\Leftrightarrow x \geq \frac{1}{\beta} \lceil \frac{1}{\alpha} n \rceil \quad \text{(since } \beta > 0\text{).}
\end{aligned}
$$

Proof Ideas-1

- First quadrant case $\alpha > 0, \beta > 0$. Now change variables to $1/\alpha, 1/\beta$.

- Using **Key Lemma**, for commutativity to hold for (new variables) $\alpha, \beta > 0$ we need the ceiling function identities:

$$\beta\lceil n\alpha\rceil = \alpha\lceil n\beta\rceil \quad \text{holds for all integer} \quad n.$$

For $n \neq 0$ rewrite this as:

$$\frac{\alpha}{\beta} = \frac{\lceil n\alpha\rceil}{\lceil n\beta\rceil}.$$

If α, β integers this relation clearly holds for all nonzero n, the floor functions have no effect.

- We have to check that if $\alpha \neq \beta$ and if they are not both integers, then commutativity fails. All we have to do is pick a good n to create a problem, if one is not integer. (Not too hard.)

Proof Ideas-2

• There is a **Key Lemma** for upper level sets of each of the other three sign patterns of α and β. (Other three quadrants). Sometimes the upper level set obtained is an open set, the finite endpoint is omitted.

• **Remark.** The discrete commuting family was used by J.-P. Cardinal (Lin. Alg. Appl. 2010) to relate the Riemann hypothesis to some interesting algebras of matrices with rational entries.

Part III. Dilated Floor Functions with Nonnegative Commutator

- The commutator function of two functions $f(x), g(x)$ is the difference of compositions

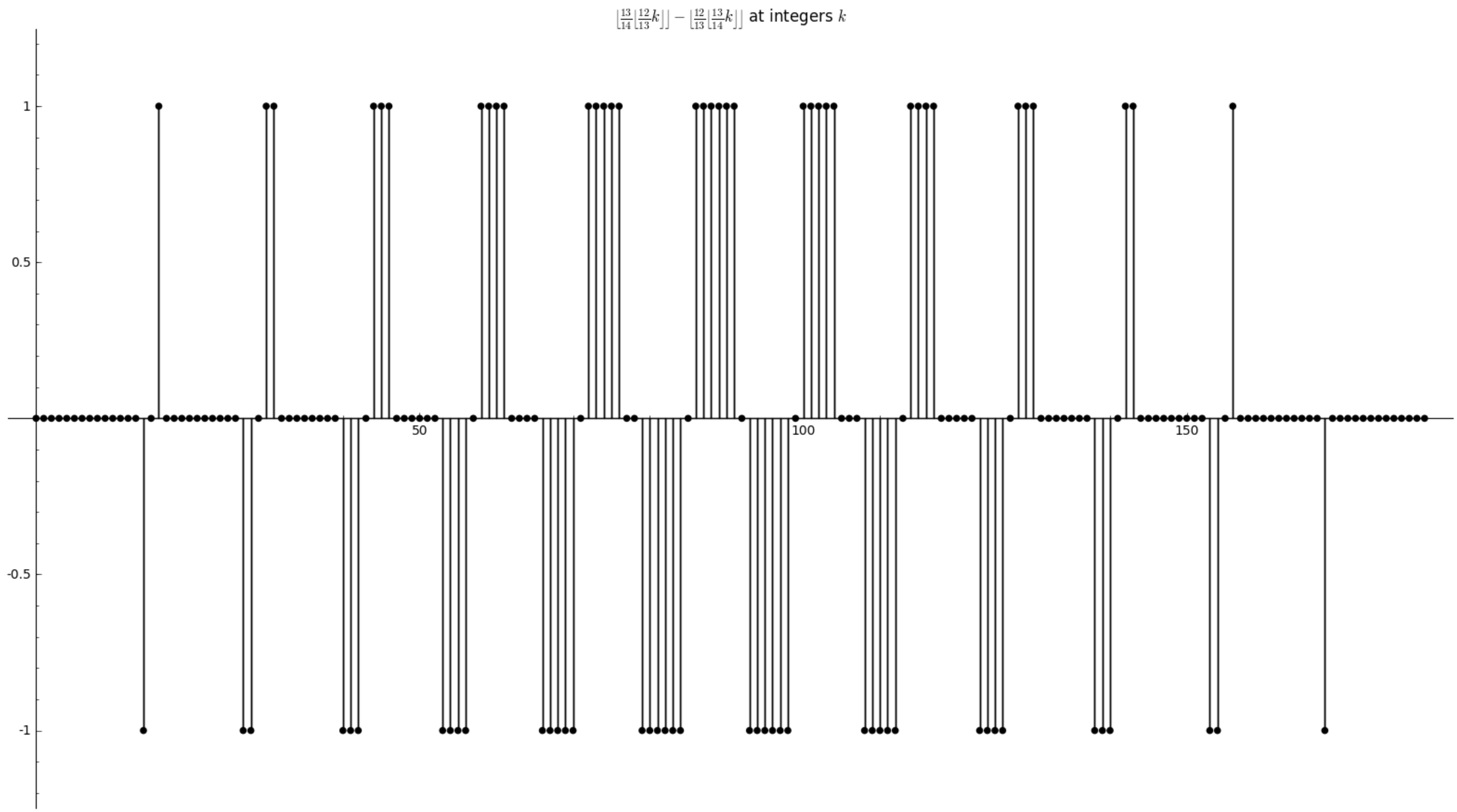
$$[f, g](x) := f(g(x)) - g(f(x)).$$

- **Question:** *Which dilated floor function pairs* (α, β) *have nonnegative commutator*

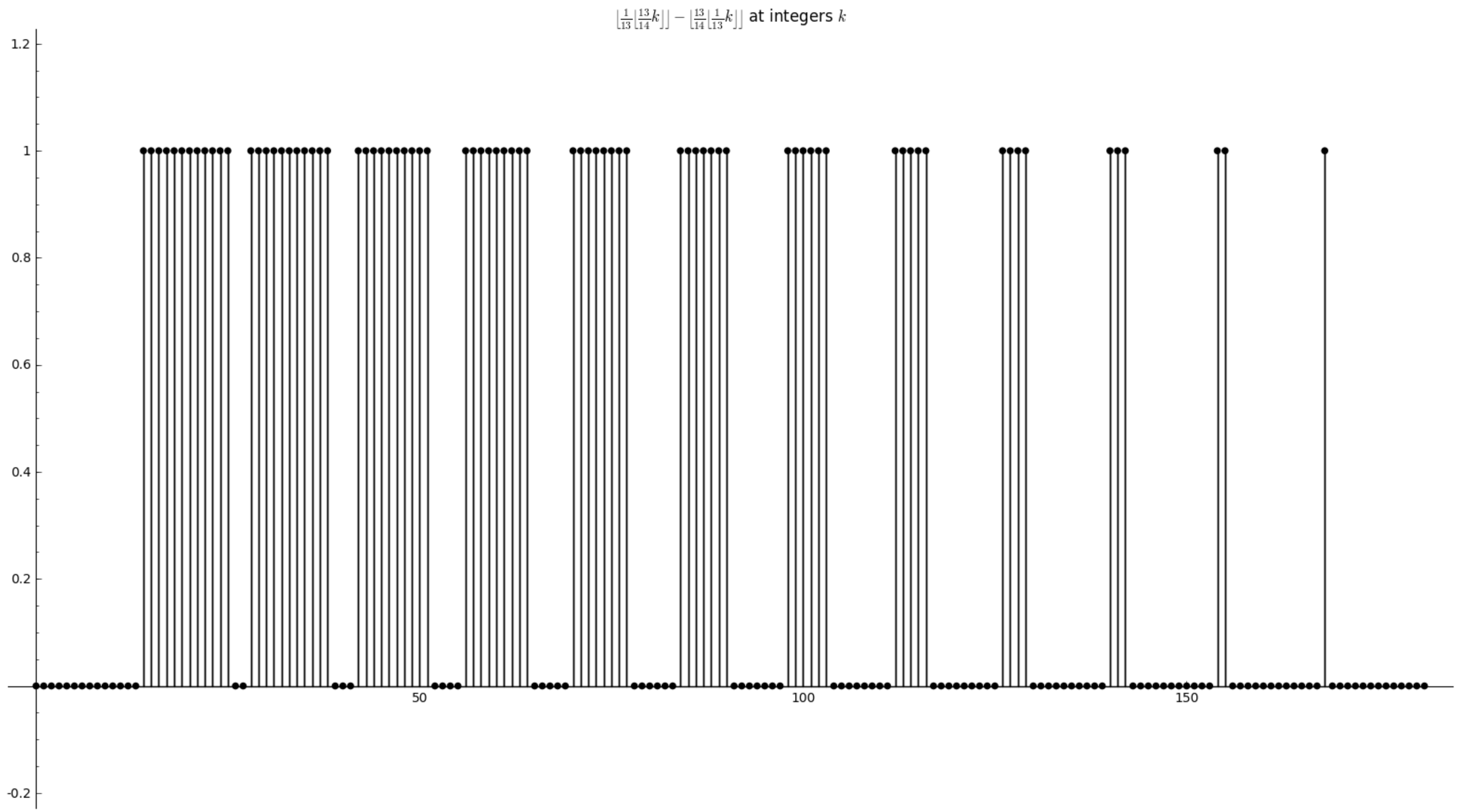
$$[f_\alpha, f_\beta] = \lfloor \alpha \lfloor \beta x \rfloor \rfloor - \lfloor \beta \lfloor \alpha x \rfloor \rfloor \geq 0 \tag{1}$$

for all real x*?*

  - We let S denote the set of all solutions (α, β) to (1).



$\alpha = \frac{13}{14}$, $\beta = \frac{12}{13}$ (table by Jon Bober)



$\alpha = \frac{1}{13}$, $\beta = \frac{13}{14}$ (table by Jon Bober)

Commutator Function-2

- Reasons to study dilated floor commutators:

  1. They measure deviation from commutativity, and are “quadratic" functions.

  2. Non-negative commutator parameters might shed light on commuting function parameters, which are the intersection of S with its reflection under the map $(\alpha, \beta) \mapsto (\beta, \alpha)$.

- For dilated floor functions the commutator function is a bounded function. It is an example of a *bounded generalized polynomial* in the sense of Bergelson and Leibman (Acta Math 2007). These arose in distribution modulo one, and in ergodic number theory.

Warmup: “Partial Commutator" Classification-1

• **Theorem.** (Partial commutator inequality classification) *The set* S_0 *of parameters* $(\alpha, \beta) \in \mathbb{R}^2$ *that satisfy the inequality*

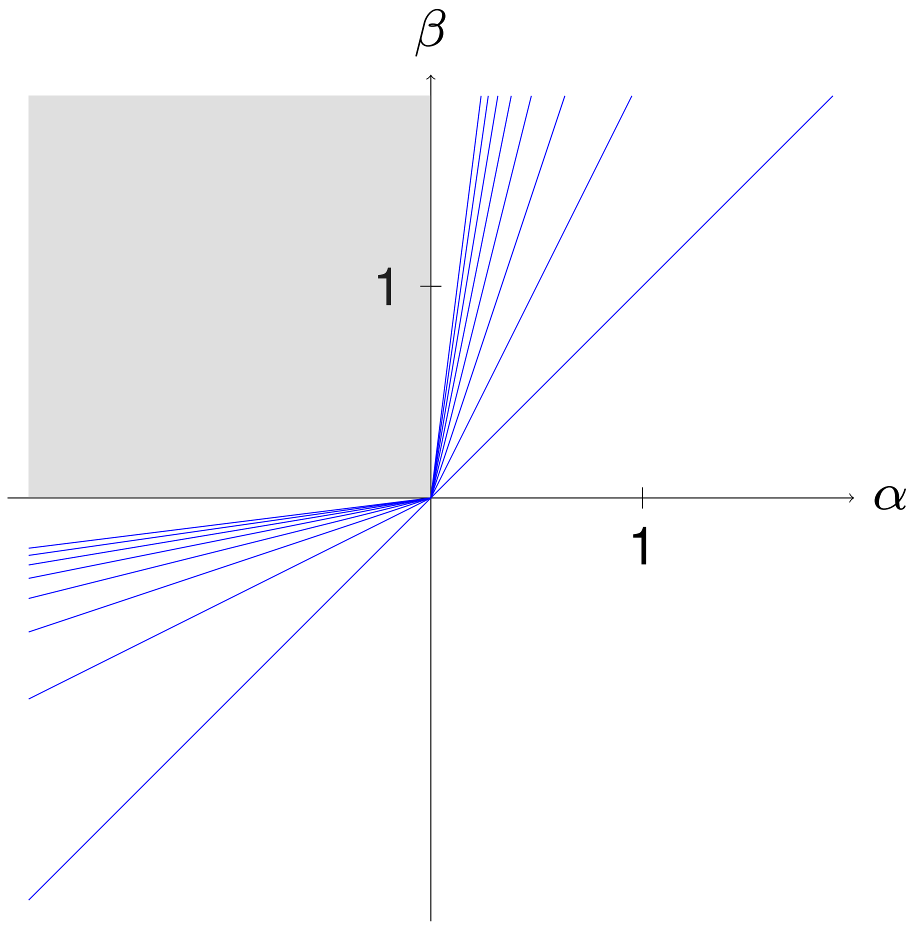
$$\alpha\lfloor \beta x\rfloor \geq \beta\lfloor \alpha x\rfloor \quad \text{for all } x \in \mathbb{R}$$

are the two coordinate axes, all points in the open second quadrant, no points in the open fourth quadrant, and:

(i) (First Quadrant) *For each integer* $m_1 \geq 1$ S_0 *contains all points with* $\alpha > 0$ *that lie on the* **oblique line** $\beta = m_1\alpha$ *of slope* m_1 *through the origin, i.e.* $\{(\alpha, m_1\alpha) : \alpha > 0\}$.

(iii) (Third quadrant) *For each integer* $m_1 \geq 1$ S_0 *contains all points with* $\alpha < 0$ *that lie on the* **oblique line** $\alpha = m_1\beta$ *of slope* $\frac{1}{m_1}$ *through the origin, i.e.* $\{(\alpha, \frac{1}{m_1}\alpha) : \alpha < 0\}$.

“Partial Commutator" Set S_0



Features of “Partial Commutator" Set S_0

• The partial commutator solution set S_0 has various symmetries.

1. The set S_0 is reflection-symmetric around the line $\alpha + \beta = 0$.
2. The set S_0 is invariant under positive dilations: If $(\alpha, \beta) \in S_0$ then $(\lambda\alpha, \lambda\beta) \in S_0$ for each real $\lambda > 0$.

• **Feature.** The partial commutator solution set S_0 lies above or on the anti-diagonal line $\alpha = \beta$ except for parts of the two coordinate axes.

• In particular, the only solutions $(\alpha, \beta) \in S_0$ that commute are the three “trivial" continuous families: $\alpha = 0$, $\beta = 0$ and $\alpha = \beta$.

Main Result: Classification of Nonnegative Commutator Parameters

• The main result classifies the structure of the set S of (α, β) for which

$$\lfloor \alpha \lfloor \beta x \rfloor \rfloor - \lfloor \beta \lfloor \alpha x \rfloor \rfloor \geq 0 \quad \text{for all} \quad x \in \mathbb{R}.$$

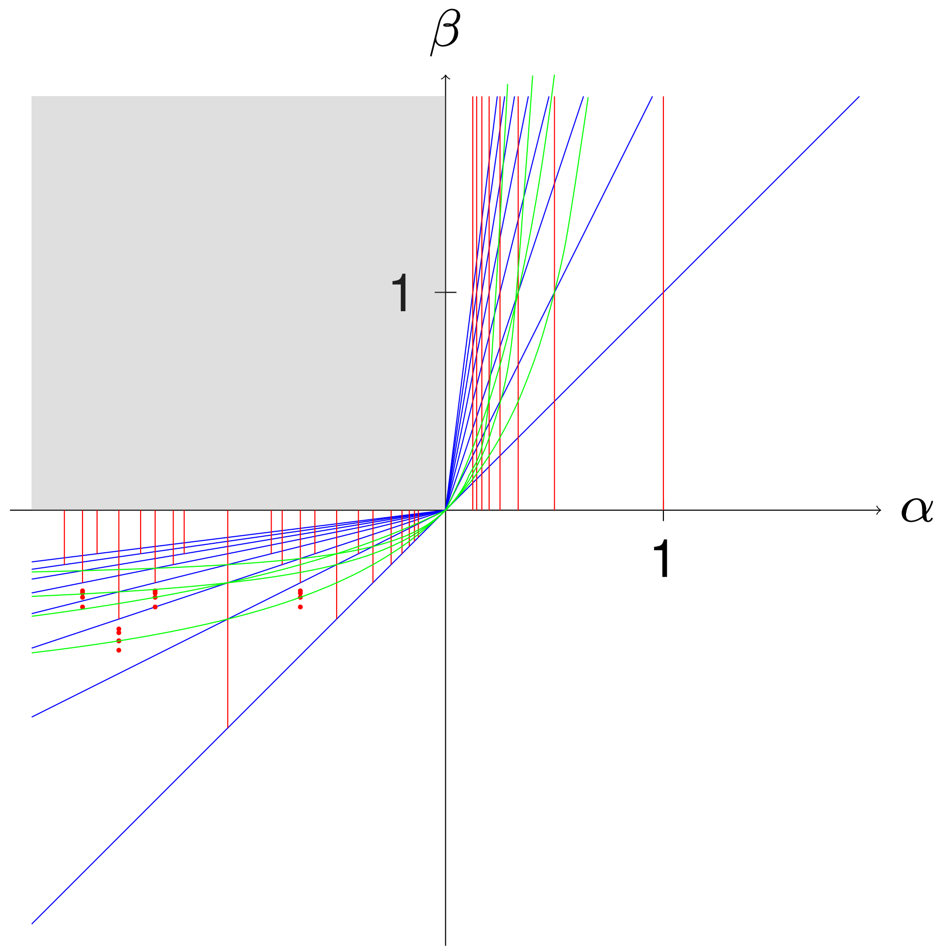
• The set S contains 2-dimensional, 1-dimensional and 0-dimensional components. These components are real semi-algebraic sets.

• The set S contains the set S_0.

• The set S has some discrete internal symmetries and also some “broken" symmetries that hold for most components but not all.

• The existence of the discrete family of solutions to the commuting dilated floor functions requires “broken" symmetries.

Classification Theorem: The Set S



Main Theorem-1: Second and Fourth Quadrant

Theorem. (Classification Theorem-1) (L & Harry Richman (2017+))

The set S of all parameters $(\alpha, \beta) \in \mathbb{R}^2$ that satisfy the inequality

$$\lfloor \alpha \lfloor \beta x \rfloor \rfloor - \lfloor \beta \lfloor \alpha x \rfloor \rfloor \geq 0 \quad \textit{for all} \quad x \in \mathbb{R}$$

consists of the coordinate axes $\{(\alpha, 0) : \alpha \in \mathbb{R}\}$ *and* $\{(0, \beta) : \beta \in \mathbb{R}\}$ *together with*

(ii) *All points in the open second quadrant,*

(iv) *No points in the open fourth quadrant,*

and the following points in the open first quadrant and third quadrant:

Main Theorem-2: First Quadrant

- **Theorem** (Classification Theorem-2)
  (i) (First Quadrant Case) *Here* $\alpha > 0$ *and* $\beta > 0$. *Points in* S *fall into three collections of one-parameter continuous families.*

(i-a) *For each integer* $m_1 \geq 1$ *all points with* $\alpha > 0$ *on the* **oblique line** $\beta = m_1\alpha$ *of slope* m_1 *through the origin, i.e.* $\{(\alpha, m_1\alpha) : \alpha > 0\}$.

(i-b) *For each integer* $m_2 \geq 1$ *all points with* $\beta > 0$ *on the* **vertical line** $\alpha = \frac{1}{m_2}$ *i.e.* $\{(\frac{1}{m_2}, \beta) : \beta > 0\}$.

(i-c) *For each pair of integers* $m_1 \geq 1$ *and* $m_2 \geq 1$, *all points with* $\beta > 0$ *on the* **rectangular hyperbola**

$$m_1\alpha\beta + m_2\alpha - \beta = 0.$$

Main Theorem-3-Third Quadrant

- **Theorem** (Classification Theorem-3)
  (iii) (Third Quadrant Case) *Here* $\alpha, \beta < 0$. *All solutions have* $|\alpha| \geq |\beta|$. *They include three collections of one parameter continuous families.*

(iii-a) *For each integer* $m_1 \geq 1$ *all points with* $\alpha < 0$ *on the* **oblique line** $\alpha = m_1 \beta$ *of slope* $\frac{1}{m_1}$ *through the origin, i.e.* $\{(\alpha, \frac{1}{m_1}\alpha) : \alpha < 0\}$.

(iii-b) *For each positive rational* $\frac{m_1}{m_2}$ *given in lowest terms, all points* $(-\frac{m_1}{m_2}, -\beta)$ *on the* **vertical line segment** $0 < \beta \leq \frac{1}{m_2}$.

(iii-c) *For each pair of integers* $m_1 \geq 1$ *and* $m_2 \geq 1$, *all points having* $\alpha < 0$ *on the* **rectangular hyperbola**

$$m_1 \alpha \beta + \alpha - m_2 \beta = 0.$$

Main Theorem-4-Third Quadrant

- **Theorem** (Classification Theorem-4)

In addition there are sporadic rational solutions *in the third quadrant.*

(iii-d) *For each positive rational* $\frac{m_1}{m_2}$ *in lowest terms satisfying* $m_1 \geq 2$*, there are infinitely many* **sporadic rational solutions** $(-\frac{m_1}{m_2}, -\beta)$*. All such sporadic solutions have* $\frac{1}{m_2} < -\beta < \frac{2}{m_2}$*, and the only limit point of such solutions is* $(-\frac{m_1}{m_2}, -\frac{1}{m_2})$*. There are no sporadic rational solutions having* $m_1 = 1$*.*

- *The set of all* **sporadic rational solutions** *having* $m_2 = 1$ *consists of all* $(\alpha, \beta) = (-m_1, -\frac{m_1 r}{m_1 r - j})$*, with integer parameters* (m_1, j, r) *having* $1 \leq j \leq m_1 - 1$*, with* $m_1 \geq 2$ *and with* $r \geq 1$*. These solutions comprise all sporadic solutions having* $\beta < -1$*.*

Classification Theorem-Discussion-1

• Compared to the “partial commutator case" or the “commuting dilations case", the first and third quadrant solutions include new continuous families of solutions. These families are parts of **straight lines** and parts of **rectangular hyperbolas**.

• **rectangular hyperbola** means: its asymptotes are parallel to the coordinate axes.

• The rectangular hyperbolas are related to *Beatty sequences*. The non-existence of first quadrant sporadic rational solutions is related to *two-dimensional Diophantine Frobenius problem*.

Interlude: Beatty Sequences

• Given a positive real number $u > 1$, its associated **Beatty sequence** is

$$\mathcal{B}(u) := \{\lfloor nu \rfloor : n \geq 1\}.$$

It is a set of positive integers.

• **"Beatty's Theorem."** *Two Beatty sequences* $\mathcal{B}(u)$ *and* $\mathcal{B}(v)$ *partition the positive integers, i.e.*

$$\mathcal{B}(u) \cup \mathcal{B}(v) = \mathbb{N}^+, \quad \mathcal{B}(u) \cap \mathcal{B}(v) = \emptyset,$$

if and only if u *and* v *lie on the rectangular hyperbola*

$$\frac{1}{u} + \frac{1}{v} = 1.$$

and u *is irrational (whence* v *is also irrational.)*

Classification Theorem-Discussion-2

• **Scaling Symmetries.** The set S is mapped into itself by some discrete families of linear maps, scaling symmetries, restricted to each quadrant. In the first quadrant, for integers $m, n \geq 1$:

$$(\alpha, \beta) \in S \quad \Rightarrow \quad (\frac{1}{n}\alpha, \frac{m}{n}\beta) \in S.$$

• **Birational Symmetries.** The set S is mapped into itself by certain birational maps, restricted to each quadrant. In the first quadrant,

$$(\alpha, \beta) \in S \quad \Leftrightarrow \quad (\frac{\alpha}{\beta}, \frac{1}{\beta}) \in S.$$

• There are additional partially broken symmetries.

Partially Broken Symmetries

• **Broken Symmetry I**. The “Partial commutator" solutions have nothing below the anti-diagonal line $\alpha = \beta$ except the α-axis and β-axis. The Classification Theorem breaks this restriction in first quadrant case (i-b). It adds some vertical lines which extend into region $\alpha > \beta$. *These extra solutions in S were necessary to get the two-parameter discrete family where dilated floor functions commute.*

• **Broken Symmetry II.** There is a partial reflection symmetry around the diagonal line $\alpha + \beta = 0$. This was perfect for the “partial commutator" case which covers oblique line cases (i-a) matching (iii-a). It also has all the hyperbolas in case (i-c) matching hyperbolas in case (iii-c). However it is broken for straight lines in case (i-b) not matching (iii-b), and the sporadic rational solutions have no counterpart at all.

S is a Closed Set

• **Corollary of Classification Theorem.** *Let* S *denote the set of all solutions* (α, β) *to* (1)*. Then* S *is a closed set in* $\mathbb{R}^2$*.*

• This fact is not obvious a priori because the functions $f_\alpha(x)$ are discontinuous in the x-variable. It was proved using the classification.

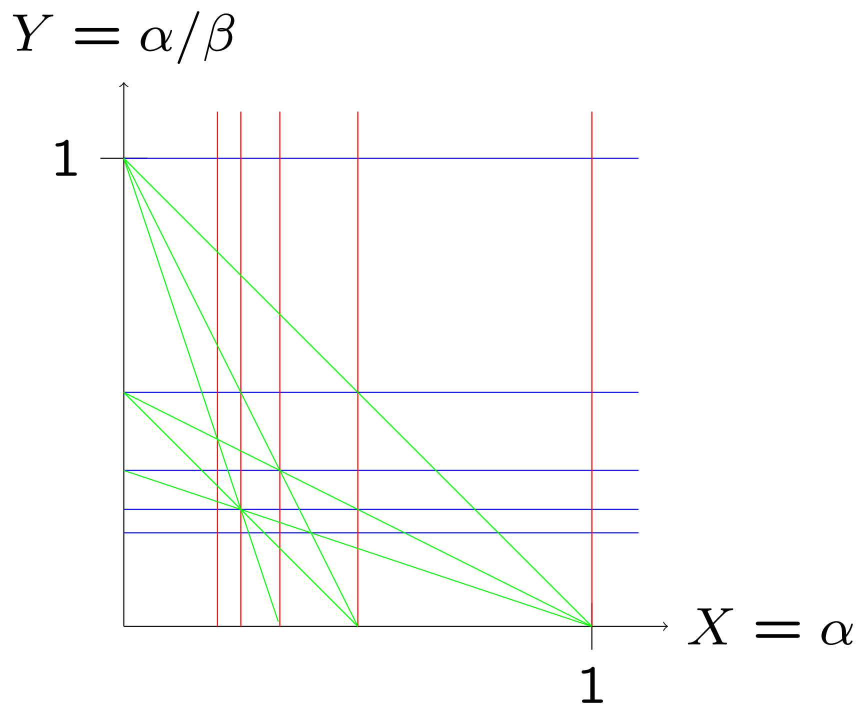
Proof Ideas-1

• Symmetries of S suggested birational changes of variables that simplify analysis. For example, using the change of variable $(X, Y) = (\alpha, \frac{\alpha}{\beta})$ makes all 1-dimensional solution curves linear. (See the next slide)

• Many equivalent conditions to (1) were found which helped analyze different parameter domains. (See two later slides)

• The connection with Beatty sequences (in a suitable coordinate system) allowed known machinery to analyze them to be used, going back to Thoralf Skolem (1957). Used topological dynamics of iterates of a point $\{k(\gamma, \delta) : k \geq 1\}$ on the unit square (modulo one), a torus.

$X - Y$ Coordinates: First Quadrant Solutions



First Quadrant Equivalences-1

The following conditions on (α, β) in the first quadrant, are successively shown to be equivalent:

(1) original inequality: $\lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha x \rfloor \rfloor$ for all $x \in \mathbb{R}$

(2) upper level set inclusions: $S_{\alpha,\beta}(n) \supseteq S_{\beta,\alpha}(n)$ for all $n \in \mathbb{Z}$

(3) rounding function inequalities $r_\alpha(n) \leq r_\beta(n)$ for all $n \in \mathbb{Z}$

(4) rescaled rounding function inequality $r_1(x) \leq r_v(x)$ for all $x \in u\mathbb{Z}$.

First Quadrant Equivalences-2

(5) disjoint residual set intersection conditions:

$$u\mathbb{Z} \cap R_v^{\pm} = \emptyset \quad \Leftrightarrow \quad R_u^{\pm} \cap R_v^{\pm} = \emptyset$$

(6) reduced Beatty sequence empty intersection condition.

(7) dual Beatty sequence empty intersection condition.

(8) All solutions with real vectors (X, Y) with $0 < X, Y < 1$ satisfy some integer dependence of form $m_1 X + m_2 Y = 1$ with nonnegative integers m_1, m_2. Solutions with $X \geq 1$ or $Y \geq 1$ are accounted for separately.

Part IV. Concluding Remarks

- Why care about this problem?

- This result found is structural about fundamental functions. The answer could not be guessed in advance.

- One-sided inequalities are potentially valuable in number theory estimates.

Thank you

Thank you for your attention!