Matroids From Hypersimplex Splits

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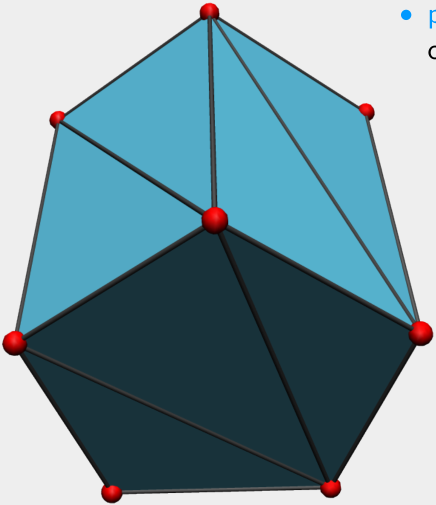
TU Berlin

Berlin, 15 December 2016

joint w/ Benjamin Schröter

Polytopes and Their Splits

Regular Subdivisions



- polytopal subdivision:
  cells meet face-to-face

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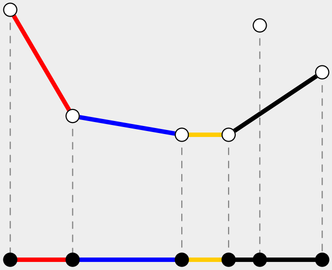

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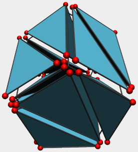
- regular: induced by weight/lifting function



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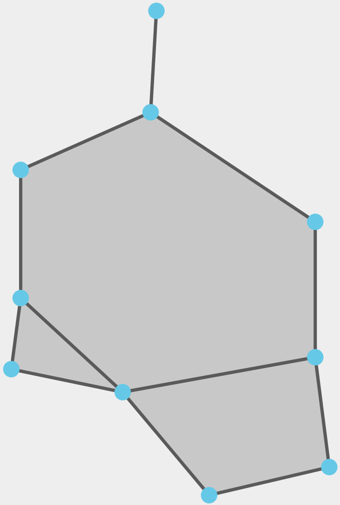
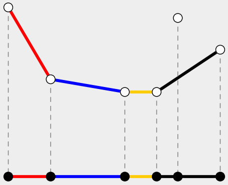
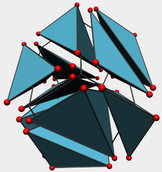


- polytopal subdivision: cells meet face-to-face
- regular: induced by weight/lifting function
- tight span = dual (polytopal) complex





Splits and Their Compatibility

Let P be a polytope.
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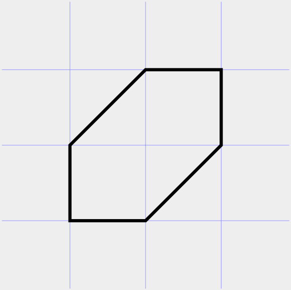

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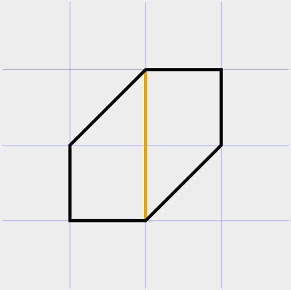

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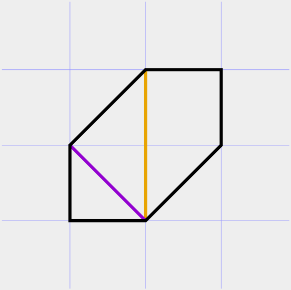

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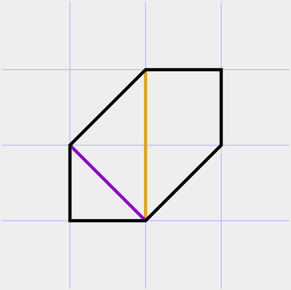

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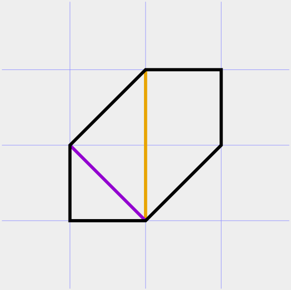

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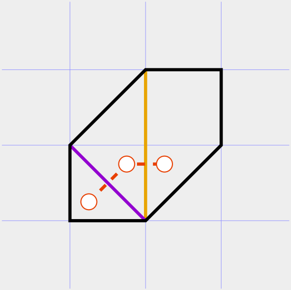

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Lemma

The tight span $\Sigma_P(\cdot)^*$ of a sum of compatible splits is a tree.

Split Decomposition

Theorem (Bandelt & Dress 1992; Hirai 2006; Herrmann & J. 2008)

Each height function w on P has a unique decomposition

$$w \;=\; w_0 + \sum_{S \text{ split of } P} \lambda_S w_S \,,$$

such that $\sum \lambda_S w_S$ weakly compatible and w_0 split prime.

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Example

$$\underbrace{(0,0,3,4,2,0)}_{w} = \underbrace{0}_{w_0} + 1\cdot\underbrace{(0,0,1,1,0,0)}_{w_S} + 1\cdot\underbrace{(0,0,2,3,2,0)}_{w_{S'}}$$

Finite Metric Spaces in Phylogenetics

Algorithmic problem

- input = finitely many DNA sequences (possibly only short)
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Key insight: think in terms of "spaces of trees"!

- Dress 1984: tight spans of finite metric spaces
  - software `SplitsTree` by Huson and Bryant
- Isbell 1963: universal properties of metric spaces
- Billera, Holmes & Vogtmann 2001
- Sturmfels & Yu 2004: polyhedral interpretation

Matroids

Matroids and Their Polytopes

Definition (matroids via bases axioms)

(d, n)-matroid $=$ subset of $\binom{[n]}{d}$ subject to an exchange condition

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Example (uniform matroid)

$U_{d,n} = \binom{[n]}{d}$

Example ($d = 2, n = 4$)

$M_5 = \{12, 13, 14, 23, 24\}$

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Definition (matroid polytope)

$P(M) =$ convex hull of char. vectors of bases of matroid M

Example (uniform matroid)

$U_{d,n} = \binom{[n]}{d}$
$P(U_{d,n}) = \Delta(d, n)$

Example ($d = 2, n = 4$)

$M_5 = \{12, 13, 14, 23, 24\}$
$P(M_5) =$ pyramid

Matroids Explained via Polytopes

Proposition (Gel′fand et al. 1987)

A polytope P is a (d, n)-matroid polytope if and only if it is a subpolytope of $\Delta(d, n)$ whose edges are parallel to $e_i - e_j$.

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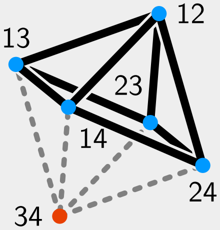
Proposition (Feichtner & Sturmfels 2005)

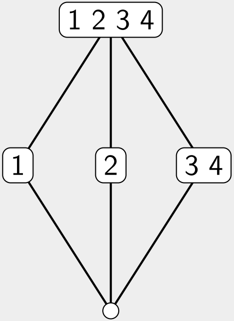
$$P(M) = \left\{ x \in \Delta(d, n) \;\middle|\; \sum_{i \in F} x_i \leq \operatorname{rank}(F),\ \textit{for } F \textit{ flat} \right\}$$

Example

$d = 2,\ n = 4,\ M_5 = \{12, 13, 14, 23, 24\}$



$P(M_5)$



lattice of flats

Example and Definition

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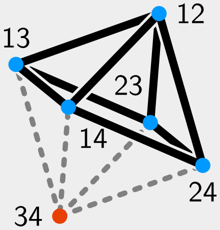

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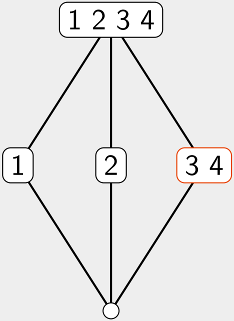


lattice of flats

Definition

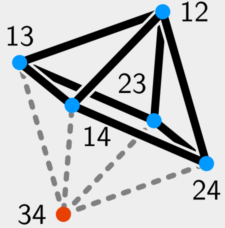
flacet = flat which is non-redundant for exterior description

Split Matroids

Definition

M split matroid $:\Longleftrightarrow$ flacets of $P(M)$ form compatible set of hypersimplex splits

- J. & Schröter 2016+: each flacet spans a split hyperplane



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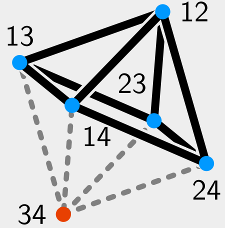

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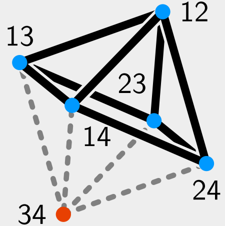
- paving matroids (and their duals) are of this type
- conjecture: asymptotically almost all matroids are paving



Percentage of Paving Matroids

$d \backslash n$	4	5	6	7	8	9	10	11	12
2	57	46	43	38	36	33	32	30	29
3	50	31	24	21	21	30	52	78	91
4	100	40	22	17	34	77	–	–	–
5		100	33	14	12	63	–	–	–
6			100	29	10	14	–	–	–
7				100	25	7	17	–	–
8					100	22	5	19	–
9						100	20	4	16
10							100	18	3
11								100	17

isomorphism classes of (d, n)-matroids:
Matsumoto, Moriyama, Imai & Bremner 2012

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3	100	100	89	75	60	52	61	80	91
4	100	100	100	75	60	82	–	–	–
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6			100	100	100	52	–	–	–
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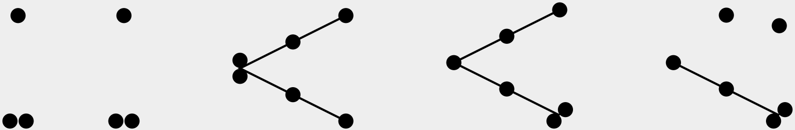
The only disconnected forbidden minor is $S_0 = M_5 \oplus M_5$, and there are precisely four connected forbidden minors:



S_1 S_2 S_3 S_4

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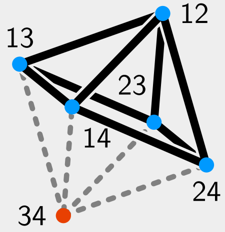
a.k.a. "valuated matroids"

Definition

Let $\pi : \binom{[n]}{d} \to \mathbb{R}$.

π (d, n)-tropical Plücker vector

$:\iff \Sigma_{\Delta(d,n)}(\pi)$ matroidal



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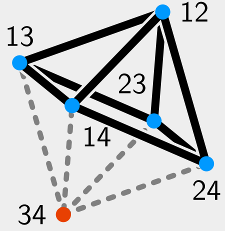

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Lemma

Each split of any matroid polytope yields matroid subdivision.

[Dress & Wenzel 1992] [Kapranov 1992] [Speyer & Sturmfels 2004]

Constructing a Class of Tropical Plücker Vectors

Let M be a (d, n)-matroid.

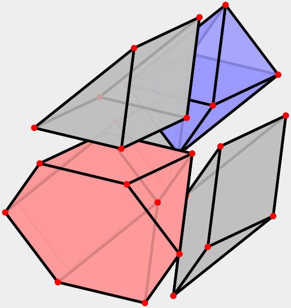
- series-free lift $\text{sf}\,M :=$ free extension followed by parallel co-extension yields $(d+1, n+2)$-matroid

Theorem (J. & Schröter 2016+)

If M is a split matroid then the map

$$\rho : \binom{[n+2]}{d+1} \to \mathbb{R}\,,\ S \mapsto d - \text{rank}_{\text{sf}\,M}(S)$$

is a tropical Plücker vector which corresponds to a most degenerate tropical linear space.



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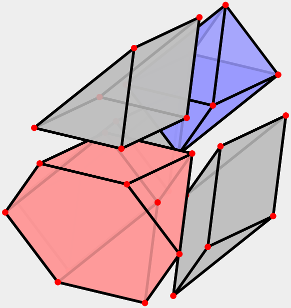

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Dressians

- Dressian $\mathrm{Dr}(d, n) :=$ subfan of secondary fan of $\Delta(d, n)$ corresponding to matroidal subdivisions
    - $\mathrm{Dr}(2, n) =$ space of metric trees with n marked leaves

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- tropical Grassmannian $\mathrm{TGr}_p(d, n) :=$ tropical variety defined by (d, n)-Plücker ideal over algebraically closed field of characteristic $p \geq 0$
  - contains tropical Plücker vectors which are realizable
  - $\mathrm{TGr}(d, n) \subset \mathrm{Dr}(d, n)$ as sets

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Corollary (J. & Schröter 2016+)

There are many rays of $\mathrm{Dr}(d, n)$ *which are not contained in* $\mathrm{TGr}_p(d, n)$ *for any* p.

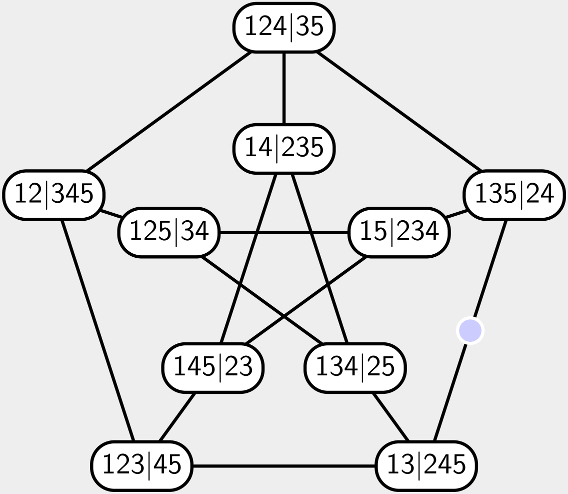
[Speyer & Sturmfels 2004] [Herrmann, J. & Speyer 2012] [Fink & Rincón 2015]

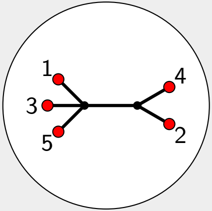
Conclusion

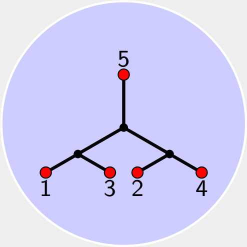
- new class of matroids, which is large
- suffices to answer previously open questions on Dressians and tropical Grassmannians
- simple characterization in terms of forbidden minors

J. & Schröter:
Matroids from hypersimplex splits, `arXiv:1607.06291`

$\mathrm{Dr}(2,5) = \mathrm{TGr}(2,5)$

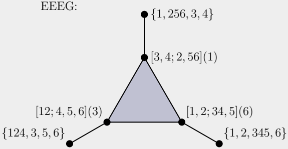




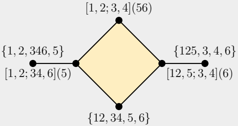


Tight Spans of Finest Matroid Subdivisions of $\Delta(3,6)$

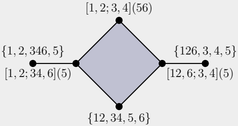
EEEG:



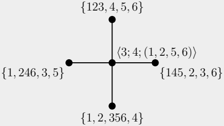
EEFF(a):



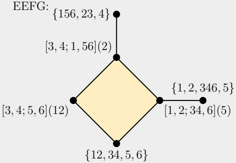
EEFF(b):



EEEE:



EEFG:



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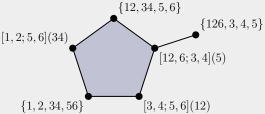

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