Real-rooted h^*-polynomials

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Unimodality and real-rootedness

Let $a_0, \ldots, a_d \geq 0$ be real numbers.

Unimodality

$a_0 \leq \cdots \leq a_i \geq \cdots \geq a_d$ for some $0 \leq i \leq d$

Unimodality and real-rootedness

Let $a_0, \ldots, a_d \geq 0$ be real numbers.

Real-rootedness of

$$a_d t^d \; + \; a_{d-1} t^{d-1} \; + \; \cdots \; + \; a_1 t \; + \; a_0$$

$$\Downarrow$$

Log-concavity

$a_i^2 \; \geq \; a_{i-1} a_{i+1}$ for all $0 \leq i \leq d$

$$\Downarrow$$

Unimodality

$a_0 \; \leq \; \cdots \; \leq \; a_i \; \geq \; \cdots \; \geq \; a_d$ for some $0 \leq i \leq d$

Interlacing polynomials

- Proof of Kadison-Singer-Problem from 1959 (Marcus, Spielman, Srivastava '15)
- Real-rootedness of independence polynomials of claw-free graphs (Chudnowski, Seymour '07)
  compatible polynomials, common interlacers
- Real-rootedness of s-Eulerian polynomials (Savage, Visontai '15)
  h^*-polynomial of s-Lecture hall polytopes are real-rooted

Further literature: Bränden '14, Fisk '08, Braun '15

Lattice zonotopes

Theorem (Schepers, Van Langenhoven '13)

The h^*-polynomial of any lattice parallelepiped is unimodal.

Theorem (Beck, J., McCullough '16)

The h^*-polynomial of any lattice zonotope is real-rooted.

Matthias Beck

Emily McCullough

Dilated lattice polytopes

Theorem (Brenti, Welker '09; Diaconis, Fulman '09; Beck, Stapledon '10)

Let P be a d-dimensional lattice polytope. Then there is an N such that the h^*-polynomial of rP has only real roots for $r \geq N$.

Conjecture (Beck, Stapledon '10)

Let P be a d-dimensional lattice polytope. Then the h^*-polynomial of rP has only distinct real-roots whenever $r \geq d$.

Theorem (Higashitani '14)

Let P be a d-dimensional lattice polytope. Then the h^*-polynomial of rP has log-concave coefficients whenever $r \geq \deg h^*(P)$.

Theorem (J. '16)

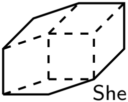
Let P be a d-dimensional lattice polytope. Then the h^*-polynomial of rP has only simple real roots whenever $r \geq \max\{\deg h^*(P) + 1, d\}$.

h^*-polynomials of *IDP*-polytopes

Conjecture (Stanley 98; Hibi, Ohsugi '06; Schepers, Van Langenhoven '13)

If P is IDP then the h^*-polynomial of P has unimodal coefficients.

- Parallelepipeds are *IDP* and zonotopes can be tiled by parallelepipeds



Shephard '74

- For all $r \geq \dim P - 1$, rP is *IDP* (Bruns, Gubeladze, Trung '97).

Outline

Interlacing polynomials

Lattice zonotopes

Dilated lattice polytopes

Interlacing polynomials

Interlacing polynomials

Definition

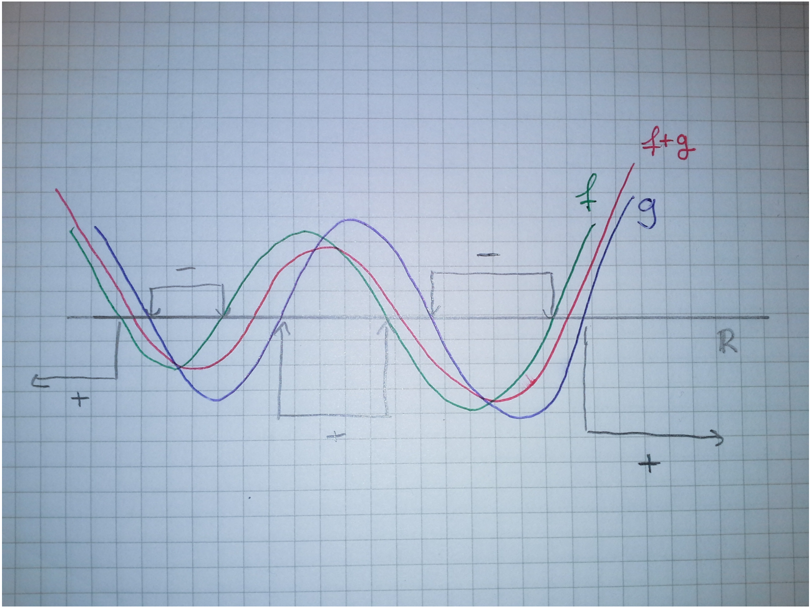
A polynomial $f = \prod_{i=1}^{m}(t - s_i)$ **interlaces** a polynomial $g = \prod_{i=1}^{n}(t - t_i)$ and we write $f \preceq g$ if

$$\cdots \leq s_2 \leq t_2 \leq s_1 \leq t_1$$

Properties

- f and g are real-rooted
- $f \preceq g$ if and only if $cf \preceq dg$ for all $c, d \neq 0$.
- $\deg f \leq \deg g \leq \deg f + 1$
- $\alpha f + \beta g$ real-rooted for all $\alpha, \beta \in \mathbb{R}$

Interlacing polynomials (schematical :-))



Polynomials with only nonpositive, real roots

Lemma (Wagner '00)

Let $f, g, h \in \mathbb{R}[t]$ be real-rooted polynomials with only nonpositive, real roots and positive leading coefficients. Then

(i) *if $f \preceq h$ and $g \preceq h$ then $f + g \preceq h$.*

(ii) *if $h \preceq f$ and $h \preceq g$ then $h \preceq f + g$.*

(iii) *$g \preceq f$ if and only if $f \preceq tg$.*

Interlacing sequences of polynomials

Definition

A sequence $f_1, \ldots, f_m$ is called interlacing if

$$f_i \preceq f_j \qquad \text{whenever } i \leq j\,.$$

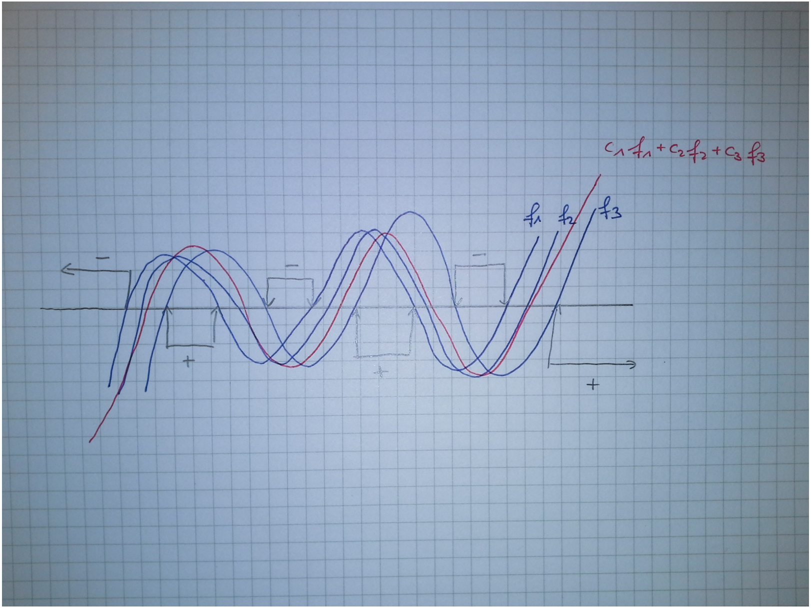
Lemma

Let $f_1, \ldots, f_m$ be an interlacing polynomials with only nonnegative coefficients. Then

$$c_1 f_1 \; + \; c_2 f_2 \; + \; \cdots \; + c_m f_m$$

is real-rooted for all $c_1, \ldots, c_m \geq 0$.

Interlacing sequences of polynomials



Constructing interlacing sequences

Proposition (Fisk '08; Savage, Visontai '15)

Let $f_1, \cdots, f_m$ be a sequence of interlacing polynomials with only negative roots and positive leading coefficients. For all $1 \leq l \leq m$ let

$$g_l = tf_1 + \cdots + tf_{l-1} + f_l + \cdots + f_m.$$

Then also $g_1, \cdots, g_m$ are interlacing, have only negative roots and positive leading coefficients.

Linear operators preserving interlacing sequences

Let $\mathcal{F}_+^n$ the collection of all interlacing sequences of polynomials with only nonnegative coefficients of length n.
When does a matrix $G = (G_{i,j}(t)) \in \mathbb{R}[t]^{m\times n}$ map $\mathcal{F}_+^n$ to $\mathcal{F}_+^m$ by $G \cdot (f_1, \dots, f_n)^T$?

Theorem (Brändén '15)

Let $G = (G_{i,j}(t)) \in \mathbb{R}[t]^{m\times n}$. Then $G \colon \mathcal{F}_+^n \to \mathcal{F}_+^m$ if and only if

(i) *$(G_{i,j}(t))$ has nonnegative entries for all $i \in [n], j \in [m]$, and*

(ii) *For all $\lambda, \mu > 0$, $1 \le i < j \le n$, $1 \le k < l \le n$*

$$(\lambda t + \mu) G_{k,j}(t) + G_{l,j}(t) \le (\lambda t + \mu) G_{k,i}(t) + G_{l,i}(t)\,.$$

Example

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ t & 1 & 1 & \cdots & 1 \\ t & t & 1 & \cdots & 1 \\ \vdots & \vdots & & & \vdots \\ t & t & \cdots & t & t \end{pmatrix} \in \mathbb{R}[x]^{(n+1)\times n}$$

(i) All entries have nonnegative coefficients ✓

Submatrices:

$$M = \begin{array}{c c} & \begin{array}{cc} i & j \end{array} \\ \begin{array}{c} k \\ l \end{array} & \begin{pmatrix} G_{k,i}(t) & G_{k,j}(t) \\ G_{l,i}(t) & G_{l,j}(t) \end{pmatrix} \end{array} \quad : \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ t & 1 \end{pmatrix} \quad \begin{pmatrix} t & 1 \\ t & t \end{pmatrix} \quad \begin{pmatrix} t & t \\ t & t \end{pmatrix}$$

(ii) $(\lambda t + \mu) G_{k,j}(t) + G_{l,j}(t) \leq (\lambda t + \mu) G_{k,i}(t) + G_{l,i}(t)$

$$(\lambda + 1)t + \mu = (\lambda t + \mu) \cdot 1 + t \leq (\lambda t + \mu) t + t = (\lambda t + \mu + 1) t \quad \checkmark$$

Lattice zonotopes

Eulerian polynomials

We call $i \in \{1, \ldots, d-1\}$ a **descent** of a permutation $\sigma \in S_d$ if $\sigma(i+1) > \sigma(i)$. The number of descents of σ is denoted by $\operatorname{des}\sigma$ and set

$$a(d,k) \;=\; |\{\sigma \in S_d : \operatorname{des}\sigma = k\}|$$

The **Eulerian polynomial** is

$$A(d,t) \;=\; \sum_{k=0}^{d-1} a(d,k)\, t^k$$

Example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \in S_3$$

123 132 213 231 312 321

$$A(3,t) \;=\; 1 + 4t + t^2$$

Theorem (Frobenius '10)

For all $d \geq 1$ the Eulerian polynomial $A(d,t)$ has only real roots.

h^*-polynomials

For every lattice polytope $P \subset \mathbb{R}^d$ let $\mathrm{E}_P(n) = |nP \cap \mathbb{Z}^d|$ be the Ehrhart polynomial of P. The h^*-polynomial $h^*(P)(t)$ of P is defined by

$$\sum_{n \geq 0} \mathrm{E}_P(n) t^n \quad = \quad \frac{h^*(P)(t)}{(1-t)^{\dim P + 1}} \, .$$

Half-open unimodular simplices
For a unimodular d-simplex Δ with facets $F_1, \ldots, F_{d+1}$

$$\mathrm{E}_\Delta(n) \; = \; \binom{n+d}{d} \; \Rightarrow \; h^*(\Delta)(t) \; = \; 1$$

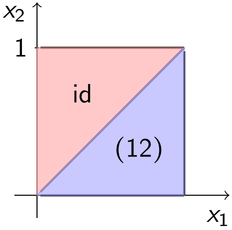
More generally, for $0 \leq i \leq d$

$$\mathrm{E}_{\Delta \setminus \bigcup_{k=1}^{i} F_k}(n) \; = \; \binom{n+d-i}{d} \; \Rightarrow \; h^*(\Delta)(t) \; = \; t^i$$

Unit cubes

Partition of unit cube $C^d = [0,1]^d$

$$C^d = \bigcup_{\sigma \in S_d} \{\mathbf{x} \in C^d : x_{\sigma(1)} \le x_{\sigma(2)} \le \cdots \le x_{\sigma(d)}\}$$

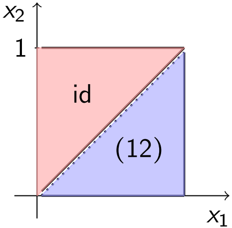


Unit cubes

Partition of unit cube $C^d = [0,1]^d$

$$C^d = \biguplus_{\sigma \in S_d} \{ \mathbf{x} \in C^d : x_{\sigma(1)} \le x_{\sigma(2)} \le \cdots \le x_{\sigma(d)},$$

$$x_{\sigma(i)} < x_{\sigma(i+1)}, \text{ if } i \text{ descent of } \sigma \}$$



Unit cubes

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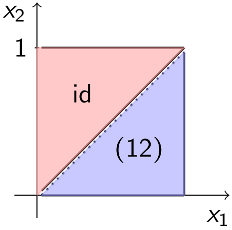


$$h^*(C^d)(t) = \sum_{\sigma \in S_d} t^{\operatorname{des}\sigma} = A(d,t)$$

Refined Eulerian polynomials

For every $j \in [d]$ we define the j-**Eulerian numbers**

$$a_j(d,k) \;=\; \left|\{\sigma \in S_d \colon \operatorname{des}\sigma = k, \sigma(1) = j\}\right|$$

and the j-**Eulerian polynomial**

$$A_j(d,k) \;=\; \sum_{k=0}^{d-1} a_j(d,k)t^k$$

Example: $d = 4, j = 2$

2134 2143 2314 2341 2413 2431

$$A(3,t) \;=\; 4t + 2t^2$$

Refined Eulerian polynomials

Lemma (Brenti, Welker '08)

For all $d \geq 1$ and all $1 \leq j \leq d+1$

$$A_j(d+1,t) \;=\; \sum_{k<i} tA_k(d,t) + \sum_{k\geq i} A_k(d,t).$$

Thus, $A_{d+1} = G \cdot A_d$, where

$$A_d = (A_1(d,t),\ldots,A_d(d,t))^T \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ t & 1 & 1 & \cdots & 1 \\ t & t & 1 & \cdots & 1 \\ \vdots & \vdots & & & \vdots \\ t & t & \cdots & t & t \end{pmatrix}$$

Theorem (Brenti, Welker '08, Savage, Visontai '15)

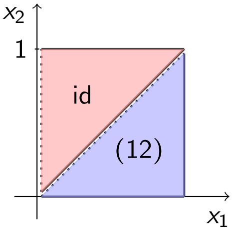
For all $1 \leq j \leq d$ the j-Eulerian polynomial $A_j(d,t)$ is real-rooted.

Half-open unit cubes

Partition of half-open unit cube $C_j^d = [0,1]^d \setminus \{x_1 = 0, \dots, x_j = 0\}$

$$C_j^d = \biguplus_{\sigma \in S_d} \{\mathbf{x} \in C_j^d : x_{\sigma(1)} \le x_{\sigma(2)} \le \cdots \le x_{\sigma(d)},$$

$$x_{\sigma(i)} < x_{\sigma(i+1)}, \text{ if } i \text{ descent of } \sigma\}$$



$h^*(C_j^d)(t) = \sum_{\sigma \in S_d} t^{\operatorname{des}_j \sigma}$ where $\operatorname{des}_j \sigma = \begin{cases} \operatorname{des}\sigma + 1 & \text{if } \sigma(1) \le j, \\ \operatorname{des}\sigma & \text{otherwise.} \end{cases}$

Refined Eulerian numbers

Claim:

$$\{\sigma \in S_d : \mathrm{des}_j\, \sigma = k\} \cong \{\sigma \in S_{d+1} : \mathrm{des}\, \sigma = k, \sigma(1) = j+1\}$$

Proof by example: $d = 5$, $j = 3$

$$24351 \;\mapsto\; 424351 \mapsto 425361$$

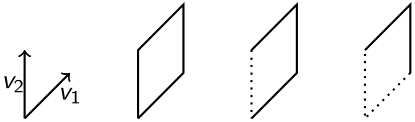
Theorem (Beck, J., McCullough '16)

$$h^*(C_j^d)(t) = A_{j+1}(d+1, t).$$

Half-open parallelepipeds

For $v_1, \ldots, v_d \in \mathbb{Z}^d$ linear independent and $I \subseteq [d]$

$$\Diamond_I(v_1, \ldots, v_d) := \left\{ \sum_{i \in [d]} \lambda_i v_i : 0 \leq \lambda_i \leq 1, 0 < \lambda_i \text{ if } i \in I \right\}$$



$I = \varnothing$ $\quad$ $I = \{1\}$ $\quad$ $I = \{1, 2\}$

Half-open parallelepipeds and zonotopes

For $K \subseteq [d]$ we denote

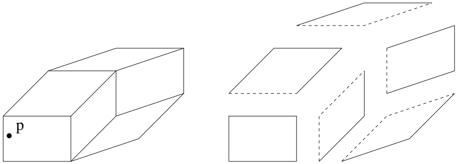
$$b(K) = |\operatorname{relint}(\lozenge(\{v_i\}_{i \in K}) \cap \mathbb{Z}^d|$$

Theorem (Beck, J., McCullough '16+)

$$h^* \left(\lozenge_I(v_1, \ldots, v_d) \right)(t) = \sum_{K \subseteq [d]} b(K) A_{|I \cup K|+1}(d+1, t).$$

In particular, the h^*-vector of every half-open parallelepiped is real-rooted.

Zonotopes



Theorem (Beck, J., McCullough '16)

The h^*-polynomial of every lattice zonotope is real-rooted.

Theorem (Beck, J., McCullough '16)

Let $d \geq 1$. Then the convex hull of the set of all h^*-polynomials of lattice zonotopes/parallelepipeds equals

$$A_1(d+1,t) + \mathbb{R}_{\geq 0} A_2(d+1,t) + \cdots + \mathbb{R}_{\geq 0} A_{d+1}(d+1,t)\,.$$

Dilated lattice polytopes

Dilation operator

For $f \in \mathbb{R}[\![t]\!]$ and an integer $r \geq 1$ there are uniquely determined $f_0, \ldots, f_{r-1} \in \mathbb{R}[\![t]\!]$ such that

$$f(t) \;=\; f_0(t^r) + tf_1(t^r) + \cdots + t^{r-1} f_{r-1}(t^r).$$

For $0 \leq i \leq r-1$ we define

$$f^{\langle r,i \rangle} = f_i.$$

Example: $r = 2$

$$1 + 3t + 5t^2 + 7t^3 + t^5$$

Then

$$f_0 = 1 + 5t \qquad f_1 = 3 + 7t + t^2$$

In particular, for all lattice polytopes P and all integers $r \geq 1$

$$\sum_{n \geq 0} \mathrm{E}_{rP}(n) t^n \;=\; \left(\sum_{n \geq 0} \mathrm{E}_P(n)(n) t^n \right)^{\langle r,0 \rangle}$$

h^*-polynomials of dilated polytopes

Lemma (Beck, Stapledon '10)

Let P be a d-dimensional lattice polytope and $r \geq 1$. Then

$$h^*(rP)(t) = \left(h^*(P)(t)(1 + t + \cdots + t^{r-1})^d\right)^{\langle r,0\rangle} .$$

Equivalently,

$$h^*(rP)(t) = h^{\langle r,0\rangle} a_d^{\langle r,0\rangle} + t\left(h^{\langle r,1\rangle} a_d^{\langle r,r-1\rangle} + \cdots + h^{\langle r,r-1\rangle} a_d^{\langle r,1\rangle}\right) ,$$

where

$$a_d^{\langle r,i\rangle}(t) := \left((1 + t + \cdots + t^{r-1})^d\right)^{\langle r,i\rangle}$$

for all $r \geq 1$ and all $0 \leq i \leq r - 1$.

Another operator preserving interlacing...

Proposition (Fisk '08)

Let f be a polynomial such that $f^{\langle r,r-1\rangle},\ldots,f^{\langle r,1\rangle},f^{\langle r,0\rangle}$ is an interlacing sequence. Let

$$g(t)=(1+t+\cdots+t^{r-1})f(t)\,.$$

Then also $g^{\langle r,r-1\rangle},\ldots,g^{\langle r,1\rangle},g^{\langle r,0\rangle}$ is an interlacing sequence.

Observation:

$$\begin{pmatrix} g^{\langle r,r-1\rangle} \\ \vdots \\ g^{\langle r,1\rangle} \\ g^{\langle r,0\rangle} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ t & 1 & 1 & \cdots & 1 \\ t & t & 1 & \cdots & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ t & t & \cdots & t & 1 \end{pmatrix} \begin{pmatrix} f^{\langle r,r-1\rangle} \\ \vdots \\ f^{\langle r,1\rangle} \\ f^{\langle r,0\rangle} \end{pmatrix}$$

Corollary

The polynomials $a_d^{\langle r,r-1\rangle}(t),\ldots,a_d^{\langle r,1\rangle}(t),a_d^{\langle r,0\rangle}(t)$ form an interlacing sequence of polynomials.

Putting the pieces together...

For all d-dimensional lattice polytopes P

$$h^*(rP)(t) = h^{\langle r,0\rangle} a_d^{\langle r,0\rangle} + t\left(h^{\langle r,1\rangle} a_d^{\langle r,r-1\rangle} + \cdots + h^{\langle r,r-1\rangle} a_d^{\langle r,1\rangle}\right)$$

Key observation: For $r > \deg h^*(P)(t)$

$$h^{\langle r,i\rangle} = h_i \geq 0 \ !$$

Theorem (J. '16)

Let P be a d-dimensional lattice polytope. Then $h^*(rP)(t)$ has only real roots whenever $r \geq \deg h^*(P)(t)$.

Concluding remarks

- Crucial: Coefficients of h^*-polynomial are nonnegative. Other applications
  - Combinatorial positive valuations
  - Hilbert series of Cohen-Macaulay domains
- Bounds are optimal
  - For Ehrhart polynomials: Only for $\deg h^*(P)(t) \leq \frac{d+1}{2}$ (using result by Batyrev and Hofscheier '10)

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Matthias Beck, Katharina Jochemko, Emily McCullough: *h*-polynomials of zonotopes, http://arxiv.org/abs/1609.08596*.

Katharina Jochemko: *On the real-rootedness of the Veronese construction for rational formal power series, http://arxiv.org/abs/1602.09139*.

Thank you