Lattice simplices of maximal dimension with a given degree

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(j.w.w. K. Kashiwabara)

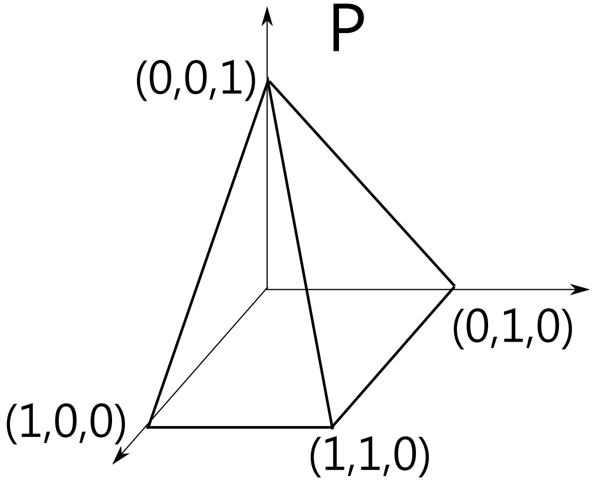
1.1. Introduction to Cayley Conjecture

Let $P \subset \mathbb{R}^d$ be a **lattice polytope**, i.e., P is a convex polytope whose vertices are the points in $\mathbb{Z}^d$.

P° : the interior of P $\quad \dim P = d$

- $\text{codeg}(P) := \min\{k : kP^\circ \cap \mathbb{Z}^d \neq \emptyset\}$
- $\deg(P) := d + 1 - \text{codeg}(P)$

Example



$\text{codeg}(P) = 3$
$\deg(P) = 3 + 1 - 3 = 1$

Why do we say $\deg(P)$ **degree** of P?

Remark For a lattice polytope $P \subset \mathbb{R}^d$, we consider the **Ehrhart series** $\sum_{n \geq 0} |nP \cap \mathbb{Z}^d| t^n$. Then this becomes a rational function of the form

$$\sum_{n \geq 0} |nP \cap \mathbb{Z}^d| t^n = \frac{h_P^*(t)}{(1-t)^{d+1}},$$

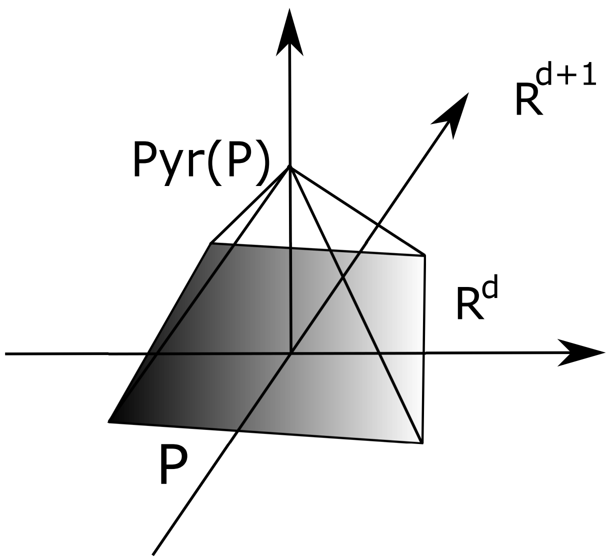
where $h_P^*(t)$ is a polynomial in t. We say that $h_P^*(t)$ is the h^***-polynomial** of P.

$$(\text{the degree of } h^*(t)) = \deg(P).$$

For a lattice polytope $P \subset \mathbb{R}^d$, a **lattice pyramid** over P is defined by

$$\mathrm{Pyr}(P) := \mathrm{conv}(\{(\alpha, 0) \in \mathbb{R}^{d+1} : \alpha \in P\} \cup \{(0, \ldots, 0, 1)\}) \subset \mathbb{R}^{d+1}.$$

Then $\dim(\mathrm{Pyr}(P)) = \dim P + 1$. In particular, those are not unimod. equiv., however...



Remark

We have $h^*_P(t) = h^*_{\mathrm{Pyr}(P)}(t)$, in particular,

$$\deg(P) = \deg(\mathrm{Pyr}(P)).$$

Motivation

We want to know **Cayley structure** of lattice polytopes.

Cayley Polytope

- $P_0, P_1, \ldots, P_\ell \subset \mathbb{R}^d$: lattice polytopes

$P_0 * P_1 * \cdots * P_\ell := \text{conv}((P_0 \times \mathbf{0}) \cup (P_1 \times \mathbf{e}_1) \cup \cdots \cup (P_\ell \times \mathbf{e}_\ell)) \subset \mathbb{R}^{d+\ell}$

We say $P_0 * \cdots * P_\ell$ is a **Cayley polytope**.

- For a lattice polytope $P \subset \mathbb{R}^{d+\ell}$, when there exist $P_0, P_1, \ldots, P_\ell \subset \mathbb{R}^d$ s.t. $P \cong P_0 * \cdots * P_\ell$, we say $P_0 * \cdots * P_\ell$ is a **Cayley decomposition** of P.

- For a lattice polytope P, let

$C(P) := \max(\{\ell + 1 : \exists P_0, \ldots, \exists P_\ell \text{ s.t.} P \cong P_0 * \cdots * P_\ell\})$.

(Strong) Cayley Conjecture (Dickenstein–Nill '12)

Let P be a lattice polytope of dimension d with degree s.

$$d > 2s \Longrightarrow C(P) \geq d + 1 - 2s.$$

(Weak) Cayley Conjecture

Let P be a lattice polytope of dimension d with degree s.

$$d > 2s \Longrightarrow C(P) \geq 2,$$

namely, P can be just decomposed into at least two polytopes.

Strong Cayley conjecture is true if

- P : smooth (Dickenstein–Nill '10)
- P : Gorenstein (DiRocco–Haase–Nill–Paffenholz '13)
- some class of $(0, 1)$-polytopes? (work in progress)

Theorem (Haase–Nill–Payne '09)

Let P be a lattice polytope of dimension d with degree s.

$$d > (s^2 + 19s - 4)/2 \Longrightarrow C(P) \geq d + 1 - (s^2 + 19s - 4)/2$$

Remark

$\exists$ **counterexample** (appear later) for **strong** Cayley Conjecture
The existence of counterexample for **weak** Cayley Conjecture might be still open.

$\longrightarrow$ I want to know $C(P)$ in order to give its "sharp" bound. I expect the bound of $C(P)$ can be given like $d + 1 -$(linear of s).

1.2. (modified) Nill's bound

On the other hand, the following theorem is known:

For $m \in \mathbb{Z}_{>0}$, let $f(m) = \sum_{\ell=0}^{\infty} \left\lfloor \frac{m}{2^\ell} \right\rfloor$.

Theorem (Nill 2008, H. 2016)

P : lattice **simplex** of dimension d with degree s

P is **NOT a lattice pyramid** $\Longrightarrow d+1 \leq f(2s) \leq 4s-1$

Moreover, $f(2s)$ is **sharp** but $f(2s) < 4s-1$ in general. (Explain later more precisely.)

Thus, it is natural to study the following problem:

Problem

Give a complete characterization of lattice simplices of dimension d with degree s satisfying $d+1 = f(2s)$.

Remark

- A complete characterization of lattice polytopes of degree 1 which are not lattice pyramids was given by Batyrev–Nill (2007).
- A complete characterization of lattice simplices of degree 2 which are not lattice pyramids was given by H.–Hofscheier (2016+).

2. Correspondence between lattice simplices and finite abelian groups

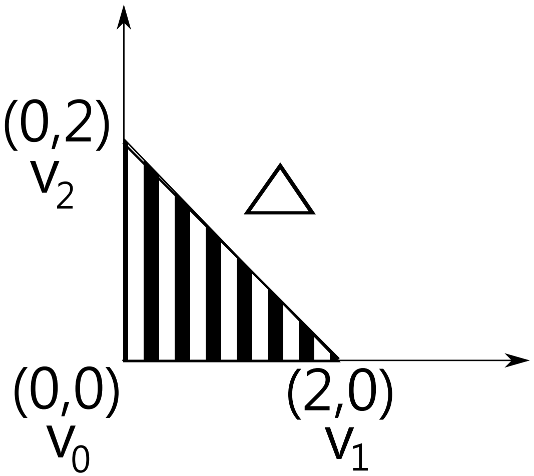
We will review the correspondence between **unimodular equvalence classes of lattice simplices** and **finite abelian groups**.

$\Delta \subset \mathbb{R}^d$: lattice simplex of dimension d

$\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathbb{Z}^d$: vertices of Δ

$$\Lambda_\Delta = \left\{ (x_0, x_1, \ldots, x_d) \in (\mathbb{R}/\mathbb{Z})^{d+1} : \sum_{i=0}^{d} x_i \mathbf{v}_i \in \mathbb{Z}^d \text{ and } \sum_{i=0}^{d} x_i \in \mathbb{Z} \right\}$$

Example



$$\Lambda_\Delta = \{\mathbf{0}, (1/2,1/2,0), (1/2,0,1/2), (0,1/2,1/2)\} = \langle (1/2,1/2,0), (1/2,0,1/2) \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$$

Λ_Δ forms a **finite abelian group**.

In this way, from a lattice simplex Δ, we can construct a finite abelian subgroup Λ_Δ of $(\mathbb{R}/\mathbb{Z})^{d+1}$.

FACTS

$$(\text{the volume of } \Delta)\cdot d! = (\text{the order of } \Lambda_\Delta)$$

$$\deg(\Delta) = \max\left\{\sum_{i=0}^{d} x_i \in \mathbb{Z}_{\geq 0} : (x_0, \ldots, x_d) \in \Lambda_\Delta, 0 \leq x_i < 1\right\}$$

$$\Delta \text{ is NOT a lattice pyramid} \iff 0 \leq \forall i \leq d,\ \exists \mathbf{x} \in \Lambda_\Delta \text{ s.t. } x_i \neq 0$$

On the other hand, from a **finite abelian subgroup** $\Lambda \subset (\mathbb{R}/\mathbb{Z})^{d+1}$ s.t.

the sum of entries of each element in Λ is an integer. $\cdots\cdots(*)$

we can construct a **lattice simplex of dim** d.

Correspondence (Batyrev–Hofscheier '13)

$$\{\textbf{lattice simplices of dim } d\}/(\textbf{unimod. equiv.})$$
$$\overset{1:1}{\longleftrightarrow}$$
$$\{\textbf{fin. abel. subgroups } \Lambda \subset (\mathbb{R}/\mathbb{Z})^{d+1} \textbf{ with } (*)\}/$$
$$(\textbf{permute of coord.})$$

Remark

$d + 1$ (dimension of Δ) $\longleftrightarrow$ $\Lambda_\Delta \subset (\mathbb{R}/\mathbb{Z})^{d+1}$

s (degree of Δ) $\longleftrightarrow$ **maximum of entry sums** of Λ_Δ

NOT a lattice pyramid $\longleftrightarrow$ $0 \leq \forall i \leq d$, $\exists \mathbf{x} \in \Lambda_\Delta$ **s.t.** $x_i \neq 0$

3. The case $d+1=4s-1$

Recall For a lattice simplex Δ of dim d with deg s, we have

$$d+1 \le f(2s) \le 4s-1.$$

Our Goal

Give a complete characterization of lattice simplices of dimension d with degree s satisfying $d+1=f(2s)$.

First, we consider the case $d+1=4s-1$, which automatically implies $d+1=f(2s)$.

Binary Simplex Codes

$C \subset (\mathbb{Z}/2\mathbb{Z})^{d+1}$: **binary simplex code**

$\Longleftrightarrow$ Binary simplex code is a binary code generated by the row vectors of the matrix $H(d+1)$

$\{\text{column vectors of } H(d+1)\} = \{{}^T(a_1, \ldots, a_{d+1}) \neq \mathbf{0} : a_i \in \{0,1\}\}$

Example

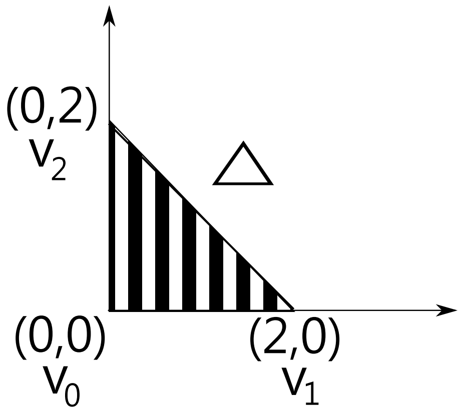
$$H(2) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad H(3) = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

binary simplex code $\Longleftrightarrow$ a dual code of **Hamming code**

We can identify **a binary code** $C \subset \mathbb{F}_2^{d+1}$ as **a finite abelian subgroup** $\Lambda \subset \{0, 1/2\}^{d+1}$. We may replace

$0 \in \mathbb{F}_2 \longleftrightarrow 0 \in \{0, 1/2\} \subset \mathbb{R}/\mathbb{Z}$ and $1 \in \mathbb{F}_2 \longleftrightarrow 1/2 \in \{0, 1/2\} \subset \mathbb{R}/\mathbb{Z}$.

Example



$$\Lambda_\Delta = \langle (1/2, 1/2, 0), (1/2, 0, 1/2) \rangle$$

comes from $H(2)$.

By Batyrev–Nill (2007), we know that this triangle is a **unique** lattice simplex of dim 2 with deg 1 s.t. $d + 1 = 4s - 1$.

Theorem (H. '16)

Δ : lattice simplex of dim d with deg s satisfying $d+1=4s-1$

Then $s=2^r$ for some $r \in \mathbb{Z}_{\geq 0}$ and

Λ_Δ comes from **binary simplex codes**.

Proposition (H. '16) $r \in \mathbb{Z}_{\geq 0}$

$\Delta(r)$: lattice simplex of dim d with deg $s=2^r$ s.t. $d+1=4s-1$

Then $C(\Delta(r)) = \dfrac{4s-1}{3} = \dfrac{d+1}{3}$.

We can see that $\Delta(r)$ becomes a counterexample for **Strong** Cayley Conjecture if $r \geq 1$.

The following looks strange and unnatural...

Question **(Modified Strong Cayley Conjecture?)**

Let P be a lattice polytope of dimension d with degree s.

$$d > \frac{8s-2}{3} \Longrightarrow C(P) \geq d+1-\frac{8s-2}{3}\ ?$$

$\Delta(r)$ satisfies this conjecture for any $r \geq 0$.

Remark $\Delta(r)$ is the **"most extremal"** among the simplices Δ of dim d with deg s s.t. $d+1 = f(2s)$.

$\longrightarrow$ MSSC is always true for any simplices?

4. The case $d+1=f(2s)$ (j.w.w. K. Kashiwabara)

Recall For a lattice simplex Δ of dim d with deg s, we have

$$d+1 \leq f(2s) \leq 4s-1.$$

We want to give a complete characterization of lattice simplices of dim d with deg s satisfying $d+1=f(2s)$ and compute $C(\Delta)$ (for checking MSCC).

By the way ...... what is $f(m)=\sum_{\ell=0}^{\infty}\left\lfloor \frac{m}{2^\ell} \right\rfloor$??

Example

$$f(2)=2+1, \quad f(3)=3+1=4,$$
$$f(4)=4+2+1, \quad f(5)=5+2+1, \quad f(6)=6+3+1, \quad f(7)=7+3+1,$$
$$f(8)=8+4+2+1, \quad f(9)=9+4+2+1 \ldots\ldots$$

Proposition

$f(m) = 2m - (\sharp \text{ of 1's for the binary expansion of } m)$

In particular, $f(m) = 2m - 1$ if and only if m is a **power of** 2.

Theorem **(again)** (H. '16)

Δ : lattice simplex of dim d with deg s satisfying $d + 1 = 4s - 1$

Then $\underset{\sim}{s = 2^r}$ for some $r \in \mathbb{Z}_{\geq 0}$ (obvious from above Prop) and Λ_Δ comes from **binary simplex codes**.

In particular, $\Lambda_\Delta \subset \{0, 1/2\}^{d+1}$.

Theorem Let Δ be a lattice simplex Δ of dim d with deg s s.t. $d + 1 = f(2s)$.

$$\Longrightarrow \Lambda_\Delta \subset \{0, 1/2\}^{d+1}$$

Proposition (H.-Kashiwabara) Let Δ be a lattice simplex of dim d with deg s s.t. $d+1 = f(2s)$. Let $2s = 2^{r_1} + \cdots + 2^{r_p}$ be the binary expansion of m, where $r_1 > \cdots > r_p \geq 1$. Then

$$r_1 + 1 \leq (\sharp \text{ of generators of } \Lambda_\Delta) \leq \sum_{i=1}^{p} r_i + p.$$

Theorem (H.-Kashiwabara) For $s \in \mathbb{Z}_{\geq 0}$, let

$$p = (\sharp \text{ of “1” in the binary expansiont of } 2s)$$

Let Δ be a lattice simplex of dim d with deg s s.t. $d+1 = f(2s)$. Assume Λ_Δ is generated by $(\lfloor \log_2 s \rfloor + 2)$ elements.

Then Λ_Δ is **uniquely determined** $\Longleftrightarrow p = 1$ or 2.

Theorem (H.-Kashiwabara) Let $2s = 2^r + 2^{r'}$ for some $r > r' \geq 1$. Let Δ be a lattice simplex of dim d with deg s s.t. $d + 1 = f(2s) = 4s - 2$. Assume Λ_Δ is generated by $(r + 1)$ elements.

Then Λ_Δ comes from the binary code generated by the row vectors of the $(r + 1) \times (4s - 2)$ matrix $(H(r + 1)\ H(r' + 1))$.

Example $$(H(3)\ H(2)) = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Theorem For any $s \geq 2$, there exists a lattice simplex Δ of dim d with deg s s.t. $d+1 = f(2s)$ satisfying $C(\Delta) < d+1-2s$, i.e., $\exists$ counterexamples of **original SCC** for $\forall s \geq 2$.

Theorem Let Δ br a lattice simplex of dim d with deg s s.t. $d+1 = f(2s)$. Then Δ always satisfies **modified SCC**.

Future Work (in progress)

- Prove **Modified Strong Cayley Conjecture** for **all lattice simplices**. $\longrightarrow$ all lattice polytopes?
- Characterize when a lattice polytope satisfies **original SCC**? (some classes of $(0,1)$-polytopes)

Danke schön.

ありがとうございます。