Tverberg's theorem over lattices and other discrete sets

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Joint work with R. La Haye, D. Rolnick, and P. Soberón

LE MENU

Tverberg-style theorems over lattices and other discrete sets

Key ideas for S-Tverberg theorems

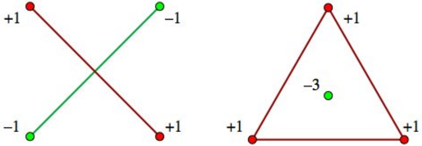
Quantitative S-Tverberg theorems

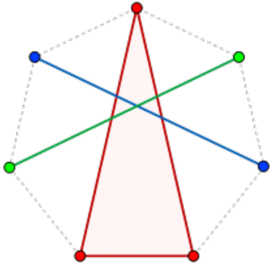
Johann Radon & Helge Tverberg:

PARTITIONING SETS OF POINTS FOR CONVEX HULLS TO INTERSECT.

Theorem (J. Radon 1920, H. Tverberg, 1966)

Let $X = \{a_1, \ldots, a_n\}$ be points in $\mathbb{R}^d$. If the number of points satisfies $n > (d+1)(m-1)$, then they can be partitioned into m disjoint parts $A_1, \ldots, A_m$ in such a way that the m convex hulls $\operatorname{conv} A_1, \ldots, \operatorname{conv} A_m$ *have a point in common.*





Remark This constant is best possible.

The $\mathbb{Z}$-Tverberg numbers

Definition

The *$\mathbb{Z}$-Tverberg number* $\mathbb{T}_{Z^d}(m)$ is the smallest positive integer such that every set of $\mathbb{T}_{\mathbb{Z}^d}(m)$ distinct integer lattice points, has a partition of the set into m sets $A_1, A_2, \ldots, A_m$ such that the intersection of their convex hulls contains at least one point of $\mathbb{Z}^d$.

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- **Theorem** (Eckhoff/ Jamison/Doignon 2000) The integer m-Tverberg number satisfies

$$2^d(m-1) < \mathbb{T}_{\mathbb{Z}^d}(m) \leq (m-1)(d+1)2^d - d - 2,$$

- for the plane with $m = 3$ is $\mathbb{T}_{\mathbb{Z}^2}(3) = 9$.
- Compare to the Tverberg over the real numbers which is 7.

Integer Radon numbers

Special case $m = 2$: an integer Radon partition is a bipartition (S, T) of a set of integer points such that the convex hulls of S and T have at least one integer point in common.

Question: How many points does one need to guarantee the existence of an **integer Radon Partition**? e.g., what is the value of $\mathbb{T}_{\mathbb{Z}^d}(2)$?

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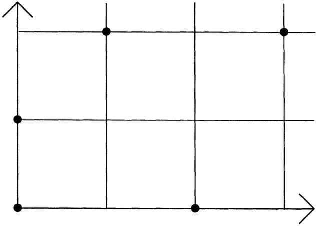
- **Theorem** (S. Onn 1991) The integer Radon number satisfies

$$5 \cdot 2^{d-2} + 1 \leq \mathbb{T}_{\mathbb{Z}^d}(2) \leq d(2^d - 1) + 3.$$

- for $d = 2$ is $\mathbb{T}_{\mathbb{Z}^d}(2) = 6$.



S-Tverberg numbers

Definition

Given a set $S \subset \mathbb{R}^d$, the *S m-Tverberg number* $\mathbb{T}_S(m)$ (if it exists) is the smallest positive integer such that among any $\mathbb{T}_S(m)$ distinct points in $S \subseteq \mathbb{R}^d$, there is a partition of them into m sets $A_1, A_2, \ldots, A_m$ such that the intersection of their convex hulls contains some point of S.

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NOTE: Original Tverberg numbers are for $S = \mathbb{R}^d$.

NOTE: When S is discrete we can also speak of a quantitative S-Tverberg numbers.

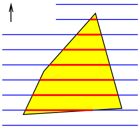
The quantitative $\mathbb{Z}$-Tverberg number

Definition

The *quantitative $\mathbb{Z}$-Tverberg number* $\mathbb{T}_{\mathbb{Z}}(m,k)$ is the smallest positive integer such that any set with $\mathbb{T}_{\mathbb{Z}}(m,k)$ distinct points in $\mathbb{Z}^d \subseteq \mathbb{R}^d$, can be partitioned m subsets $A_1, A_2, \ldots, A_m$ where the intersection of their convex hulls contains **at least** k points of $\mathbb{Z}^d$.

Interesting Examples of $S \subset \mathbb{R}^d$

- A natural (non-discrete) is $S = \mathbb{R}^p \times Z^q$.



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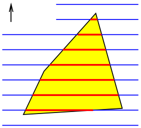
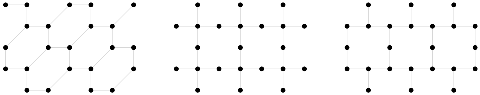


- Let L be a lattice in $\mathbb{R}^d$ and $L_1, \ldots, L_p$ sublattices of L. Set $S = L \setminus (L_1 \cup \cdots \cup L_p)$.



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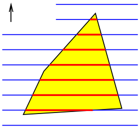

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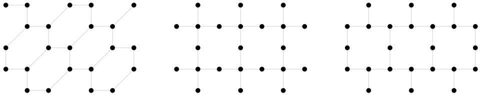
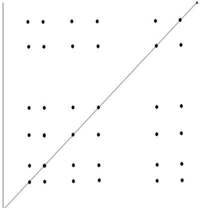


- Let $S =$ Primes $\times$ Primes



OUR RESULTS

Improved $\mathbb{Z}$-Tverberg numbers

Corollary (DL, La Haye, Rolnick, Soberón, 2015)

The following bound on the Tverberg number exist:

$$\mathbb{T}_{\mathbb{Z}^d}(m) \leq (m-1)d2^d + 1.$$

Discrete quantitative $\mathbb{Z}$-Tverberg

Corollary (DL, La Haye, Rolnick, Soberón, 2015)

Let $c(d,k) = \lceil 2(k+1)/3 \rceil 2^d - 2\lceil 2(k+1)/3 \rceil + 2$.

The quantitative $\mathbb{Z}$*-Tverberg number* $\mathbb{T}_{\mathbb{Z}}(m,k)$ *over the integer lattice* $\mathbb{Z}^d$ *is bounded by*

$$\mathbb{T}_{\mathbb{Z}^d}(m,k) \leq c(d,k)(m-1)kd + k.$$

S-Tverberg number for interesting families

Corollary

The following Tverberg numbers $\mathbb{T}_S(m)$ exist and are bounded as follows:

1. *When $S = \mathbb{Z}^{d-a} \times \mathbb{R}^a$, we have* $\mathbb{T}_S(m) \leq (m-1)d(2^{d-a}(a+1)) + 1.$

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2. *Let L be a lattice in $\mathbb{R}^d$ of rank r and let $L_1, \dots, L_p$ be p sublattices of L. Call $S = L \setminus (L_1 \cup \cdots \cup L_p)$ the difference of lattices. The quantitative Tverberg number satisfies*

$$\mathbb{T}_S(m,k) \leq \left(2^{p+1}k + 1\right)^r (m-1)kd + k.$$

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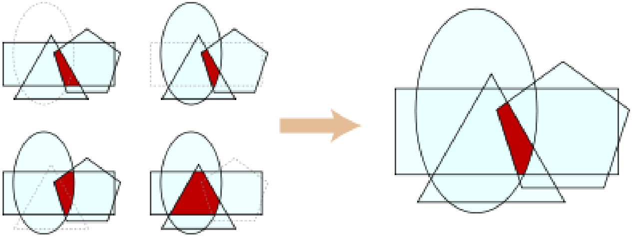
EXAMPLE Let L', L'' be sublattices of a lattice $L \subset \mathbb{R}^d$, then, if $S = L \setminus (L' \cup L'')$, the Tverberg number satisfies $\mathbb{T}_S(m) \leq 6(m-1)d2^d + 1$.

KEY IDEAS

HELLY's THEOREM (1914)

Given a finite family H of convex sets in $\mathbb{R}^d$. If every d+1 of its elements have a common intersection point, then all elements in H has a non-empty intersection.



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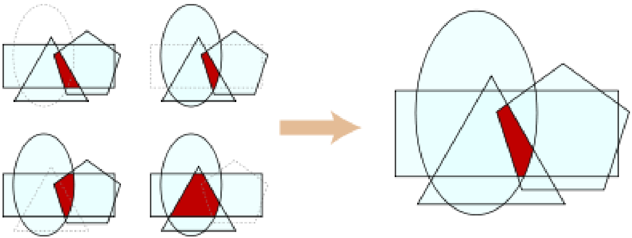


For $S \subseteq \mathbb{R}^d$ let $\mathcal{K}_S = \{S \cap K : K \subseteq \mathbb{R}^d \text{ is convex}\}$. The *S-Helly number* $h(S)$ is the smallest natural number satisfying

$$\forall i_1, \ldots, i_{h(S)} \in [m] : F_{i_1} \cap \cdots \cap F_{i_{h(S)}} \neq \emptyset \quad \Longrightarrow \quad F_1 \cap \cdots \cap F_m \neq \emptyset \tag{1}$$

for all $m \in \mathbb{N}$ and $F_1, \ldots, F_m \in \mathcal{K}_S$. Else $h(S) := \infty$.

Integer Helly theorem

Jean-Paul Doignon (1973)

DOIGNON'S theorem

Given a finite collection D of convex sets in $\mathbb{R}^d$, the sets in D have a common point with integer coordinates if every 2^d of its elements do.

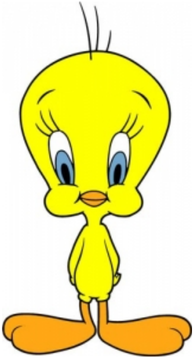
Mixed Integer version of Helly's theorem

Hoffman (1979) Averkov & Weismantel (2012)

Theorem

Given a finite collection D of convex sets in $\mathbb{Z}^{d-k} \times \mathbb{R}^k$, if every $2^{d-k}(k+1)$ of its elements contain a mixed integer point in the intersection, then all the sets in D have a common point with mixed integer coordinates.

CENTRAL POINT THEOREMS



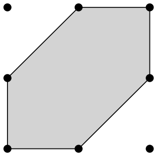
- There exist a point p in such that no matter which line one traces passing through p leaves at least $\frac{1}{3}$ of the area of the body in each side!

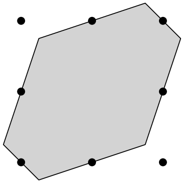
S-Helly numbers

Lemma (Hoffman (1979), Averkov & Weismantel)

Assume $S \subset \mathbb{R}^d$ is discrete, then the Helly number of S, $h(S)$, is equal to the following two numbers:

1. *The supremum $f(S)$ of the number of facets of an S-facet-polytope.*
2. *The supremum $g(S)$ of the number of vertices of an S-vertex-polytope.*





NOTE With **Deborah Oliveros, Edgardo Roldán-Pensado** we obtained several S-Helly numbers.

MAIN THEOREM

S-Tverberg number must exists when the S-Helly number exists!!

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S-Tverberg number must exists when the S-Helly number exists!!

Theorem

Suppose that $S \subseteq \mathbb{R}^d$ is such that $h(S)$ exists. (In particular, S need not be discrete.) Then, the S-Tverberg number exists too and satisfies

$$\mathbb{T}_S(m) \leq (m-1)d \cdot h(S) + 1.$$

Sketch of proof of S-Tverberg

- **An central point theorem for S:** Let $A \subseteq S$ be a set with at least $(m-1)dh(S)+1$ points. Then there exist a point $p \in S$ such that every closed halfspace $p \in H^+$ satisfies $|H^+ \cup S| \geq (m-1)d+1$.
- Consider the family of convex sets

$$\mathcal{F} = \{F | F \subset A, |F| = (m-1)d(h(S)-1)+1\}.$$

- For any $\mathcal{G}$ subfamily of $\mathcal{F}$ with cardinality $h(S)$ the number of points in $A \setminus F$ for any $F \in \mathcal{F}$ is

$$(m-1)dh(S)+1-(m-1)d(h(S)-1)-1 = (m-1)d.$$

  The number of points in $A \setminus \bigcup \mathcal{G}$ is at most $(m-1)dh(S)$.
- Since this is less than $|A|$, $\bigcap \mathcal{G}$ must contain an element of S. By the definition of $h(S)$, $\bigcap \mathcal{F}$ contains a point p in S.
- This is the desired p. Otherwise, there would be at least $(m-1)d(h(S)-1)+1$ in its complement.
- That would contradict the fact that every set in $\mathcal{F}$ contains p.

- **Claim** We can find m disjoint (*simplicial*) subsets $A_1, A_2, \ldots, A_m$ of A that contain p.
- Suppose we have constructed $A_1, A_2, \ldots, A_j$ for some $j < m$, and each of them is a simplex that contains p in its relative interior.
- If the convex hull of $S_j = S \setminus (A_1 \cup \cdots \cup A_j)$ contains p, then we can find a simplex A_{j+1} that contains p in its relative interior. Otherwise, there is a hyperplane H that contains p that leaves S_j in one of its open half-spaces.
- Then $H^+ \cap S_j = \emptyset$. However, since $|H^+ \cap A_i| \leq d$,

$$(m-1)d \geq jd \geq |H^+ \cap A_1| + \cdots + |H^+ \cap A_j| = |H^+ \cap S| \geq (m-1)d+1,$$

a contradiction.

QUANTITATIVE CONVEXITY

Circa October 2014....we began working on

arXiv:1503.06116

Title: Quantitative Tverberg, Helly, & Carathéodory theorems
Authors: J. A. De Loera, R. N. La Haye, D. Rolnick, P. Soberón
Categories: math.MG Metric Geometry (math.CO Combinatorics)
Comments: 33 pages

Abstract: This paper presents sixteen quantitative versions of the classic Tverberg, Helly, & Caratheodory theorems in combinatorial convexity. Our results include measurable or enumerable information in the hypothesis and the conclusion. Typical measurements include the volume, the diameter, or the number of points in a lattice.

Owner: Pablo Sober\'on
Version 1: Fri, 20 Mar 2015 15:36:31 GMT
Version 2: Mon, 23 Mar 2015 13:20:31 GMT
Version 3: Thu, 2 Apr 2015 01:44:28 GMT

A Quantitative Integer-Helly theorem

Theorem (Iskander Aliev, JDL, Quentin Louveaux, 2013)

- *For d, k non-negative integers, there exists a constant $c(d,k)$, determined by k and dimension d, such that*

  For any finite family $(X_i)_{i\in\Lambda}$ of convex sets in $\mathbb{R}^d$, if

  $$|\bigcap_{i\in\Lambda} X_i \cap \mathbb{Z}^d| = k,$$

  then there is a subfamily, of size no more than $c(d,k)$, with exactly the same integer points in its intersection.
- *For d, k non-negative integers*

  $$c(d,k) \le \lceil 2(k+1)/3 \rceil 2^d - 2\lceil 2(k+1)/3 \rceil + 2$$

Discrete quantitative S-Tverberg

Given $S \subset \mathbb{R}^d$, let $h_S(k)$ be the smallest integer t such that whenever finitely many convex sets have exactly k common points in S, there exist at most t of these sets that already have exactly k common points in S.

Theorem

Let $S \subseteq \mathbb{R}^d$ be discrete set with finite quantitative Helly number $h_S(k)$. Let m, k be integers with $m, k \geq 1$. Then, we have

$$\mathbb{T}_S(m, k) \leq h_S(k)(m-1)kd + k.$$

Quantitative S-Helly numbers

We generalized Hoffman's theorem to provide a way to bound the quantitative S-Helly

Definition

A set $P \subset S$ is *k-hollow* with respect to S if

$$\left|(\mathrm{conv}(P) \setminus V(\mathrm{conv}(P))) \cap S\right| < k,$$

where $V(K)$ is the vertex set of K.

Lemma

Let $S \subset \mathbb{R}^d$ be a discrete set. The quantitative S Helly number is bounded above by the cardinality of the largest k-hollow set with respect to S.

Lemma

Let L be a lattice in $\mathbb{R}^d$ of rank r and let $L_1, \dots, L_p$ be p sublattices of L. Call $S = L \setminus (L_1 \cup \cdots \cup L_p)$ the difference of lattices. The quantitative S-Helly number $h_S(k)$ exists and is bounded above by $\left(2^{p+1}k + 1\right)^r$.

Improvements

- Chestnut et al. 2015 improved our theorem, for fixed d, to $c(d,k) = O(k(loglogk)(logk)^{-1/3})$ and gave lower bound $c(d,k) = \Omega(k^{(n-1)/(n+1)})$
- Averkov et al. 2016 gave have a different new combinatorial description of $\mathbb{H}_S(k)$ in terms of polytopes with vertices in S. Consequences:
  - They strengthen our bound of $c(d,k)$ by a constant factor
  - For fix d showed that $c(d,k) = \Theta(k^{(d-1)/(d+1)})$ holds.
  - Determined the exact values of $c(d,k)$ for all $k \leq 4$.

Upcoming work

Theorem (DL, Nabil Mustafa, Frédéric Meunier)

The integer m-Tverberg number in the plane equals $\mathbb{T}_{\mathbb{Z}^2}(m) \leq 4m - 3 + k$, *where* k *is the smallest non-negative integer that makes the number of points congruent with zero modulo* m, *thus for* $m > 3$, *this is the same as* $4m$.

Doignon (unpublished) There is point set with $\mathbb{T}_{\mathbb{Z}^2}(m) > 4m - 4$

Coming up in 2017: Integer Tverberg and stochastic optimization!

OPEN PROBLEM: What is the exact value for $11 \leq \mathbb{T}_{\mathbb{Z}^3}(2) \leq 17$ (K. Bezdek + A. Blokhuis 2003)?

OPEN PROBLEM: Find better upper bounds, lower bounds!

OPEN PROBLEM: Algorithms to find integer Tverberg partitions.

OPEN PROBLEM: Is there a Helly number for $S = (PRIMES)^2$?

THANK YOU!
DANKE!
GRACIAS!