

Minkowski-type theorems for lattice polytopes

Einstein Workshop on Lattice Polytopes

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- $\mathcal{P}(\mathbb{Z}^d)$ can be split into subfamilies

$$\mathcal{P}^d(k) := \left\{ P \in \mathcal{P}(\mathbb{Z}^d) : |\text{int}(P) \cap \mathbb{Z}^d| = k \right\} \quad \text{with } k = 0, 1, 2, \dots$$

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- Lattice simplices will play a particular role. So, we also introduce:

$$\mathcal{S}^d(k) := \left\{ S \in \mathcal{P}^d(k) : S \text{ simplex} \right\}$$

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$k \geq 1$: regular case

Extensively studied:

Scott 1976 Zaks & Perles & Wills 1982

Hensley 1983

Rabinovitz 1989

Lagarias & Ziegler 1991

Borisov & Borisov 1992

Borisov 2000

Pikhurko 2001

Conrads 2002

Nill 2005

Nill 2007

Haase & Schicho 2009

Kasprzyk 2009

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A. 2012

Kasprzyk 2013

Ambro 2014

A. & Krümpelmann & Nill 2015

Balletti & Kasprzyk & Nill 2016

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Theorem (Hensley 1983)

$$\text{vol}(P) \leq 2^{2^{O(d \log d)}} \quad \forall P \in \mathcal{P}^d(1)$$

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Zaks & Perles & Wills just give an example and do calculations.

Idea behind the example of Zaks & Perles & Wills

Split the unity into Egyptian fractions (Egyptian partition)

$$1 = \frac{1}{a_1} + \dots + \frac{1}{a_d} \quad (a_1, \dots, a_d \in \mathbb{N})$$

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- Let's construct Egyptian partitions of 1 involving a huge denominator a_d .

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- It grows fast!

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s_d	d
2	1
3	2
7	3
43	4
1807	5
3263443	6
10650056950807	7
113423713055421844361000443	8
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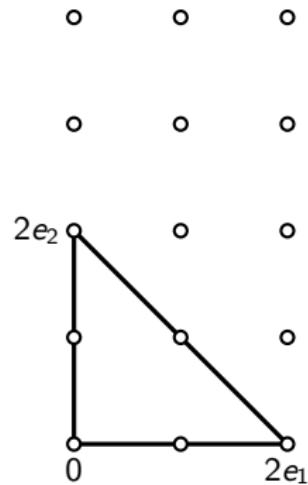
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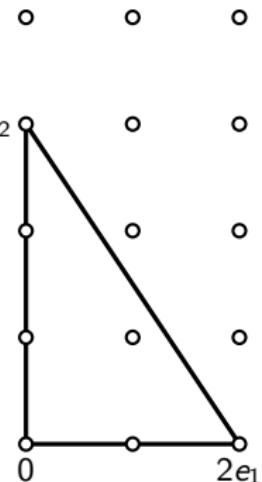
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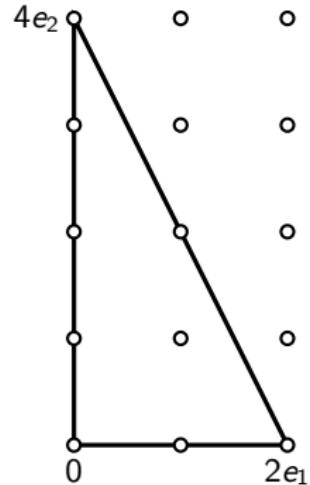
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has a huge volume (the *Zaks & Perles & Wills simplex*)

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- \Rightarrow

$$\text{conv}(0, s_1 e_1, \dots, s_{d-1} e_{d-1}, 2(s_d - 1) e_d) \in \mathcal{S}^d(1)$$

has a huge volume (the *Zaks & Perles & Wills simplex*)

Note

$$\text{conv}(0, s_1 e_1, \dots, s_{d-1} e_{d-1}, s_d e_d) \in \mathcal{S}^d(1)$$

also has some interesting properties (the *Hensley simplex*)

Our approach to upper bounds

Deal with $\mathcal{S}^d(1)$ *first* and *then* with $\mathcal{P}^d(1)$

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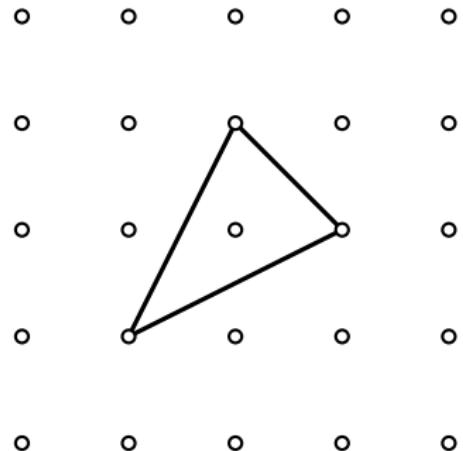
$$\text{int}(K) \cap \mathbb{Z}^d = \{0\} \quad \Rightarrow \quad \text{vol}(K) \leq 2^d.$$

Message

Under the 0-symmetry assumption, the complicated condition $\text{int}(K) \cap \mathbb{Z}^d = \{0\}$ is relaxed to an easier condition $\text{vol}(K) \leq 2^d$.

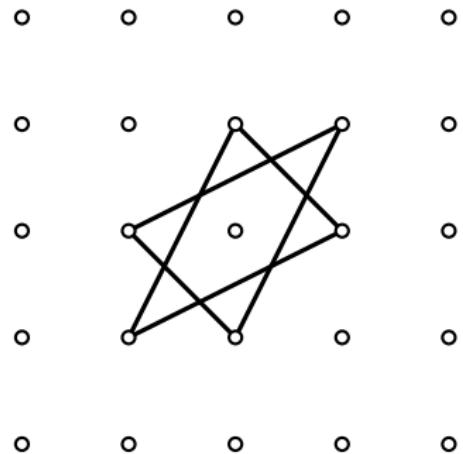
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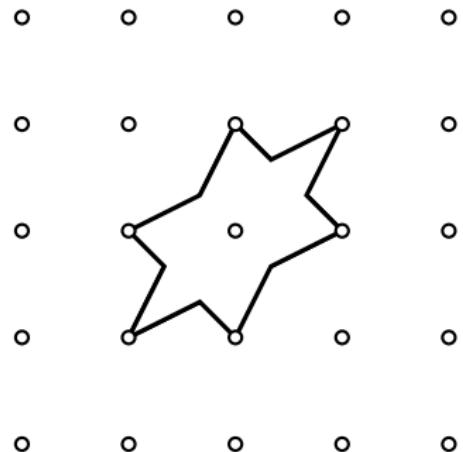
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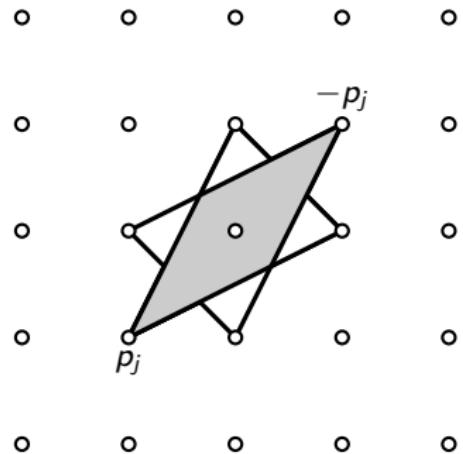
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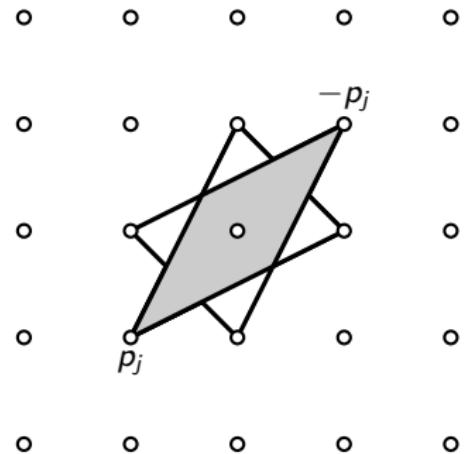
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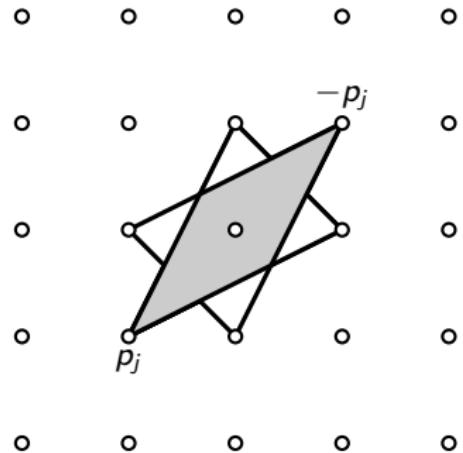
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- Minkowski $\Rightarrow \text{vol}(P) \leq 2^d$.



$\text{vol}(P)$ vs. $\text{vol}(S)$

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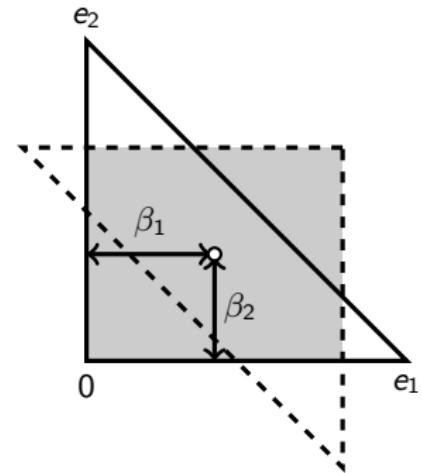
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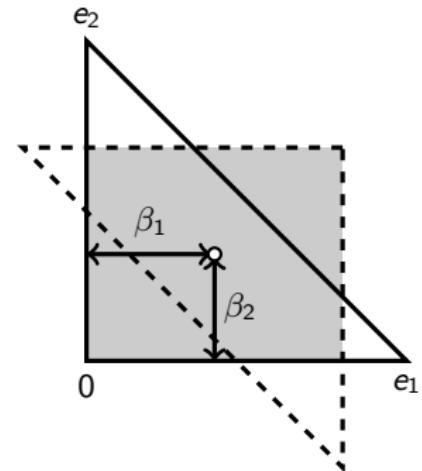
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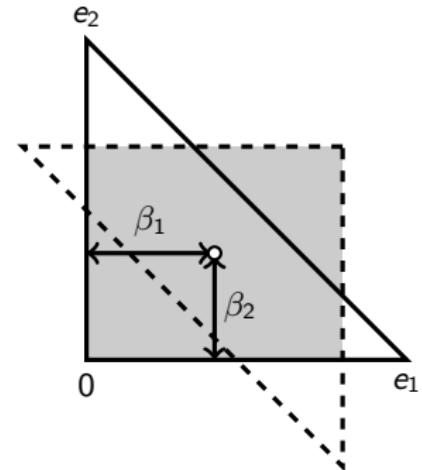
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$$\text{vol}(S) \leq \frac{1}{d! \prod_{i \in I} \beta_i}, \quad \text{whenever } |I| = d$$



Bounding the volume of $S \in \mathcal{S}^d(1)$

- Upper bounds on $S \iff$ Lower bounds on $\beta_1, \dots, \beta_{d+1}$

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- Argument of Hensley: a geometric component + an analytic component (estimates).

Product-sum inequalities

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Lemma (Determinant lemma)

Let $A \in \mathbb{R}^{d \times d}$ be an invertible matrix such that

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Message

The complicated set of invertible A such that $\|Az\|_\infty < 1$ has only one integer solution $z = 0$ is relaxed to the easier set

$$\left\{ A \in \mathbb{R}^{d \times d} : |\det(A)| \geq 1 \right\}.$$

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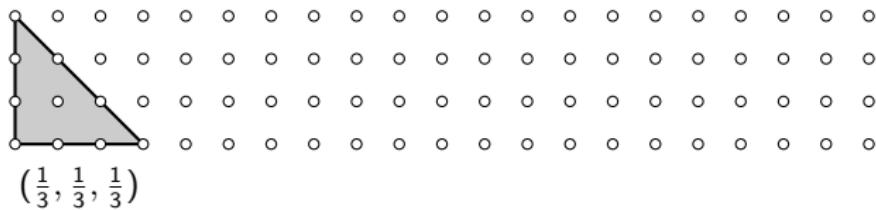
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The same message

The complicated set of the vectors $(\beta_1, \dots, \beta_{d+1})$ of barycentric coordinates of interior lattice points of $S \in \mathcal{S}^d(1)$ is relaxed to a set of barycentric coordinates described by the product-sum inequalities.

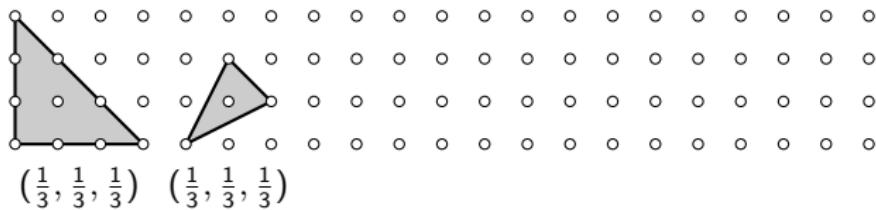
Example

$d = 2$



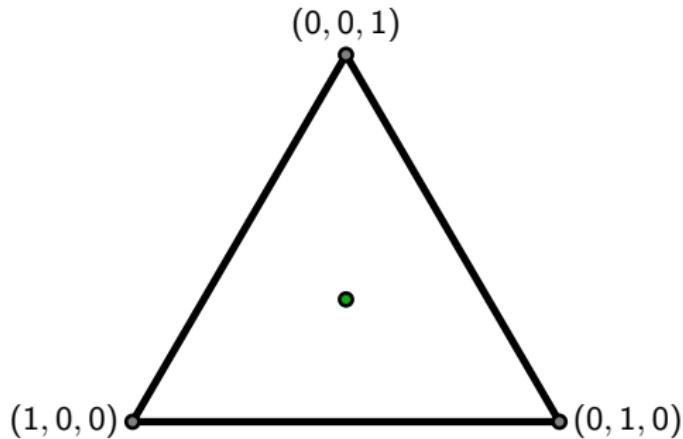
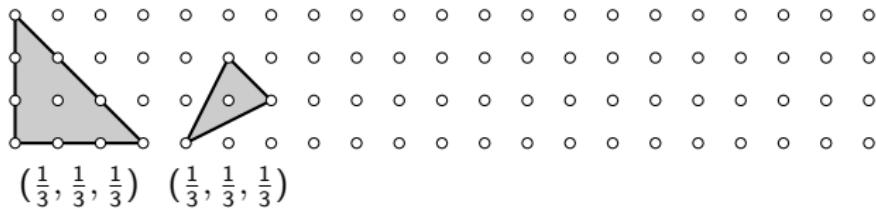
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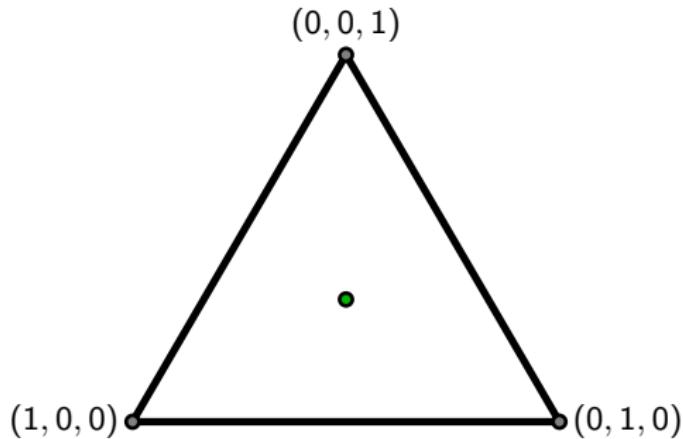
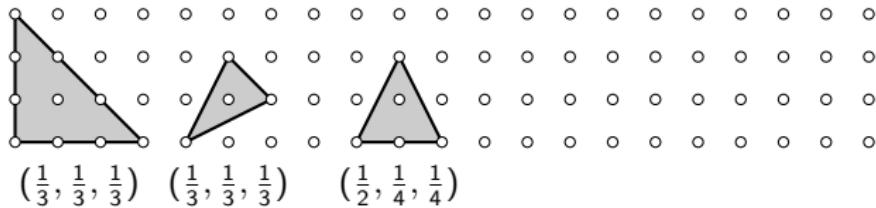
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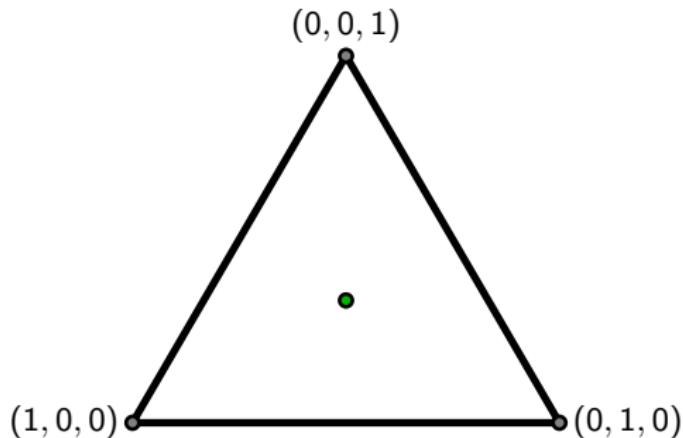
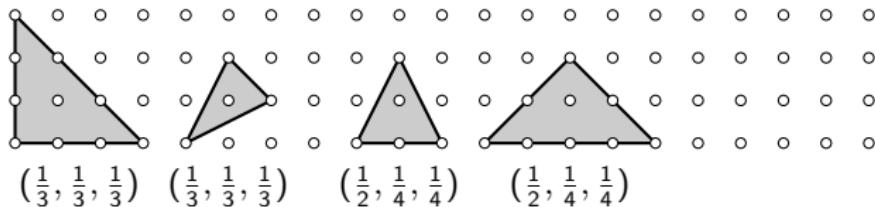
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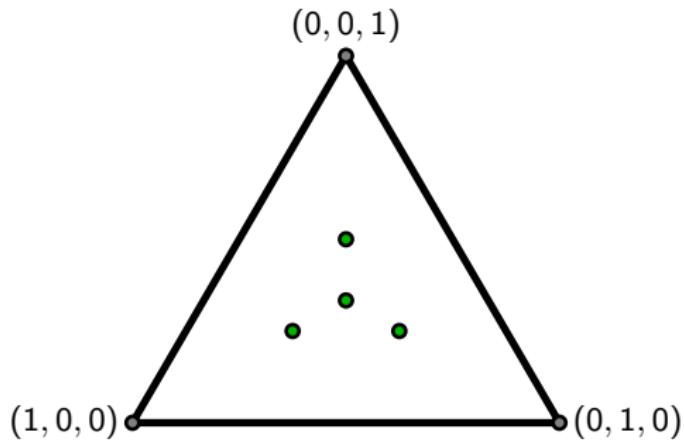
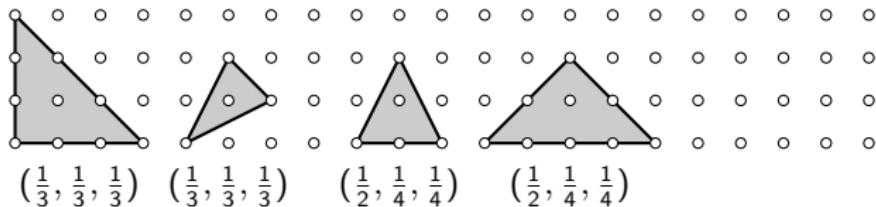
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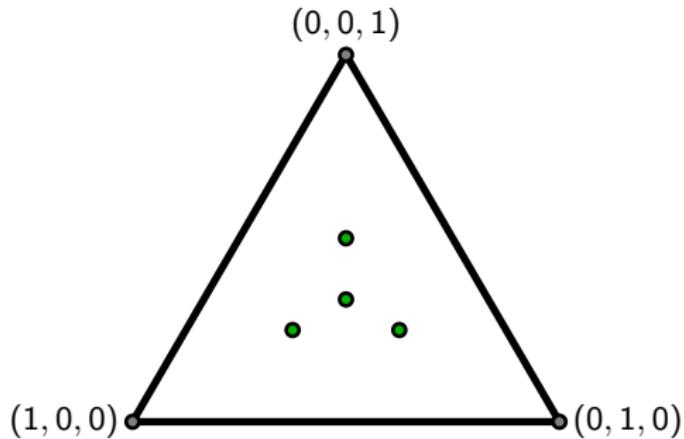
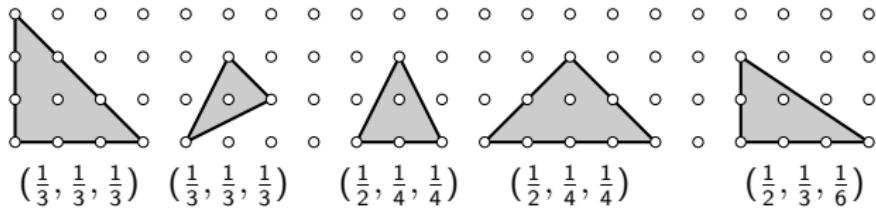
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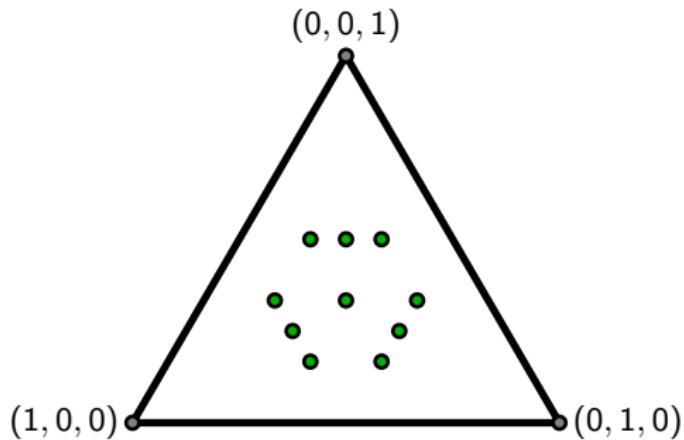
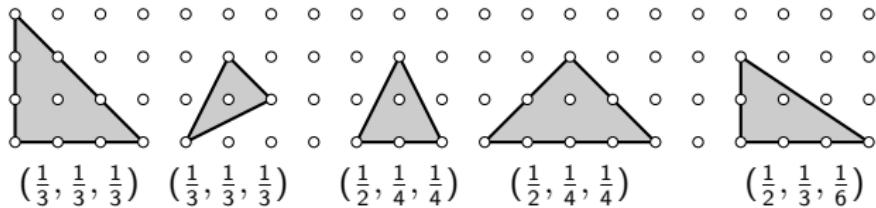
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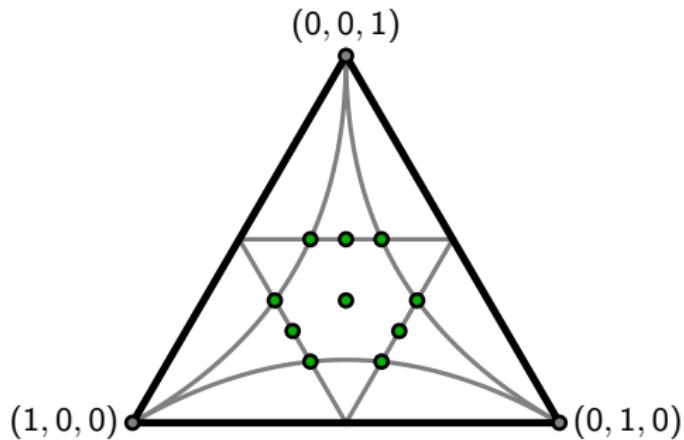
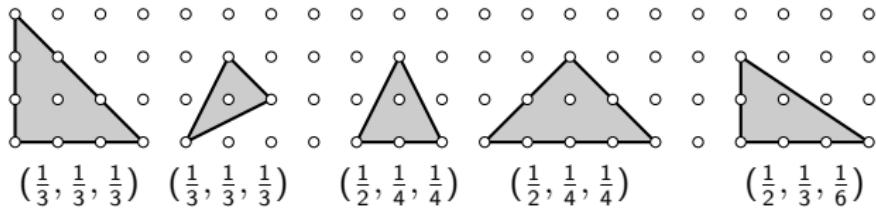
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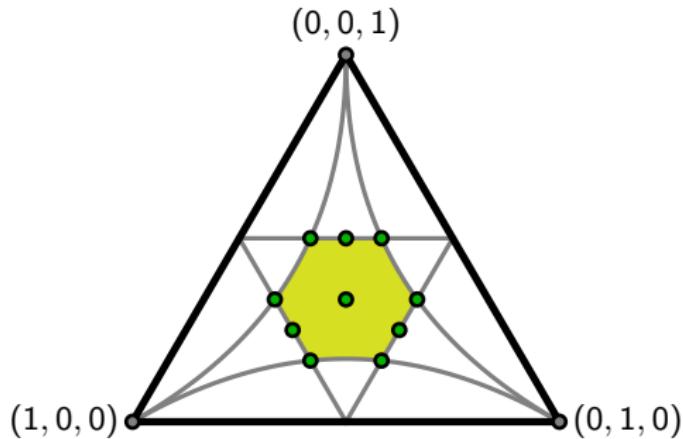
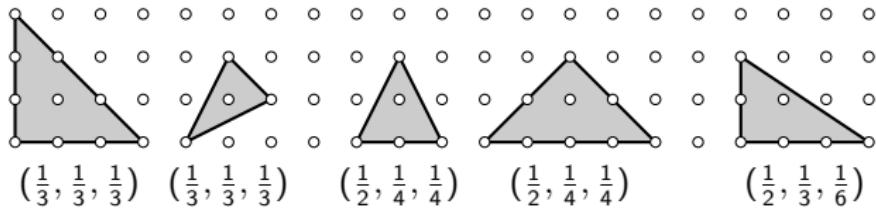
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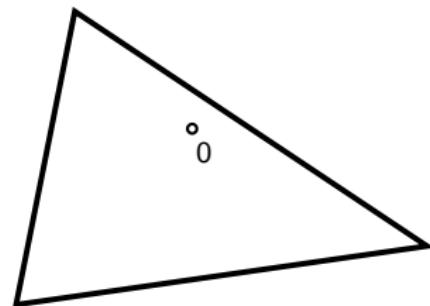
Sketch of the proof (1)

- An arbitrary rational point in $\text{aff}(p_1, \dots, p_i)$

$$r := \frac{m_1}{m} p_1 + \dots + \frac{m_i}{m} p_i$$

$m_1, \dots, m_i \in \mathbb{Z}, \ m \in \mathbb{N}$

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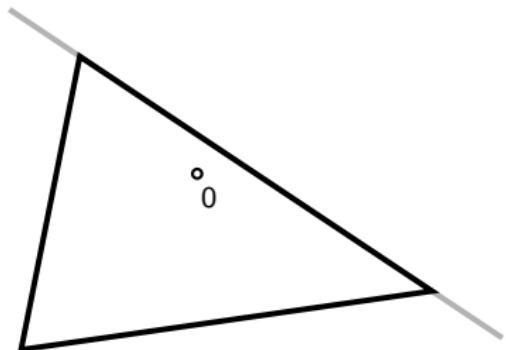
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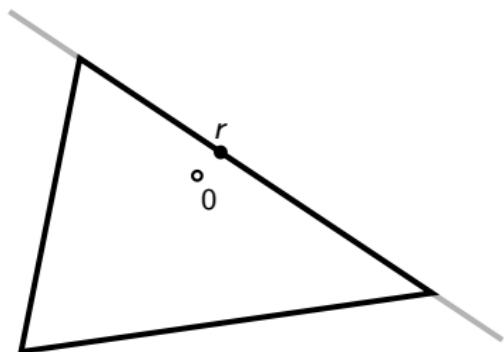
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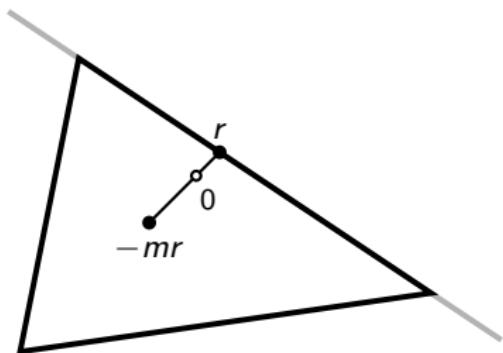
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- Equivalence for $-mr \in \mathbb{Z}^n$:

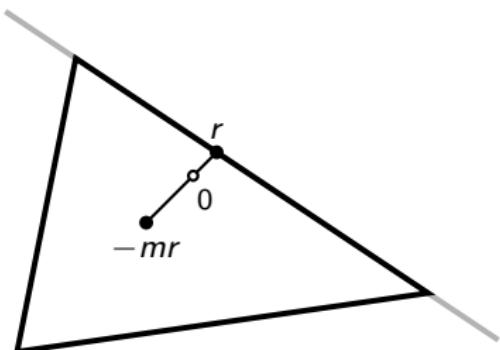
$$-mr \in \text{int}(S)$$

\Leftrightarrow

$$(m+1)\beta_1 - m_1 > 0$$

\dots

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- Quite a good bound

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$$\text{vol}(S) \leq \frac{(d+1)^{2^d - 1}}{d!} \quad \forall S \in \mathcal{S}^d(1)$$

- Quite a good bound (but not the best one)

- To improve volume bounds we need a better bound for $\beta_1 \cdots \beta_d$ in the estimate

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$$\beta_1 \cdots \beta_j \leq \beta_{j+1} + \dots + \beta_{d+1} \quad (\text{product-sum})$$

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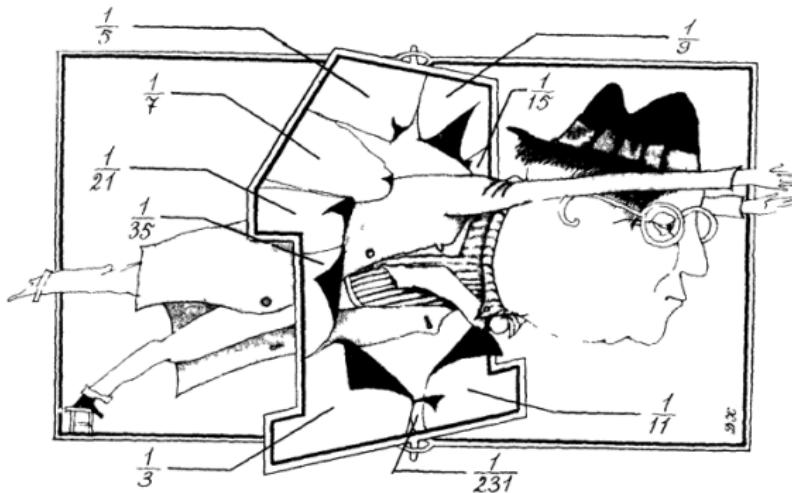
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- Minimize $\beta_1 \cdots \beta_d$.



A very similar problem considered by Izhboldin & Kurliandchik, Kvant, 1987



Математический круассан

Разбиение единицы

О. Т. ИЖВОЛДИН. Л. Д. КУРЛЯНДЧИК

Существует много задач, связанных с суммами чисел, обратных натуральным. Неоднократно обращался к этой теме и наш журнал: в прошлом году в «Задачнике «Кванта» были помещены две такие задачи — М989 и М1010 (в «Кванте» № 6 и № 10). Решение первой из них было основано на

$$\frac{1}{r_1} + \frac{1}{r_2} + \dots + \frac{1}{r_n}.$$

*
По существу здесь требуется найти наилучшее приближение единицы снизу суммами дробей вида $1/r$ (кстати, такие числа носят название аликовитых, или египетских, дробей). Эта задача принадлежит американскому математику Келлогу. В 1915 году профессор Кармайкл, которому Келлог сообщил о ней, привел ее в своей книге «Диофантов анализ». В 1922 году она была решена американским математиком Картесиусом. Математическая

Learning from Izhboldin & Kurliandchik (1)

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$$\frac{1}{a_1} + \dots + \frac{1}{a_{d+1}} = 1$$

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- These are exactly the product-sum inequalities!

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- Minimize β_{d+1} subject to (product-sum,normalization,ordering)



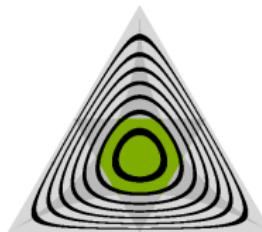
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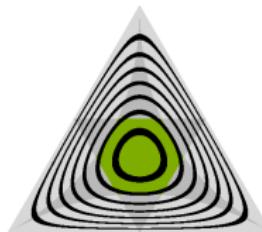
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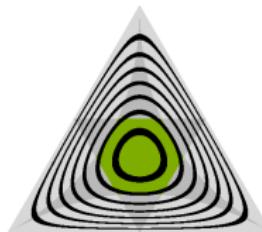


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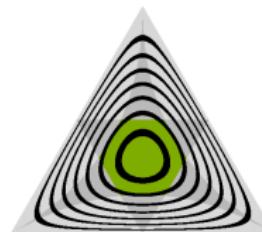
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- Proof: perturbation argument.
- Minimization of β_{d+1} is solved as well, because

$$\beta_1 \dots \beta_{d+1} \leq \beta_{d+1}^2$$

and for the above optimal solution the equality is attained.



A conjecture of Hensley

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- \Rightarrow Hensley's conjecture is true

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Theorem (A. & Krümpelmann & Nill 2015)

For $d \geq 3$, the simplex

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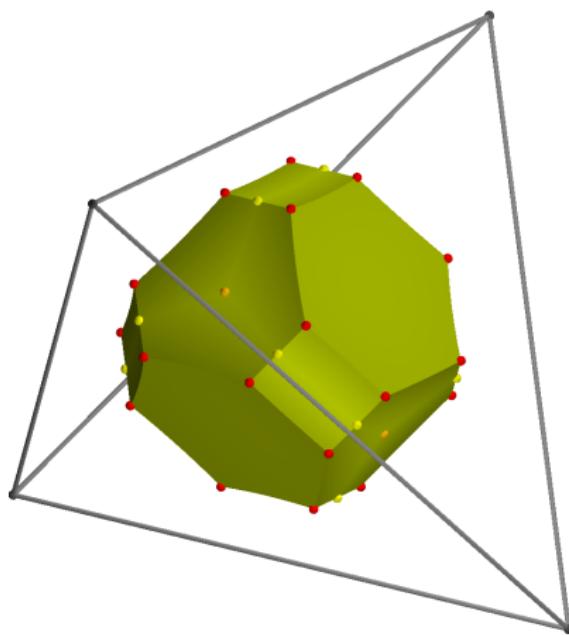
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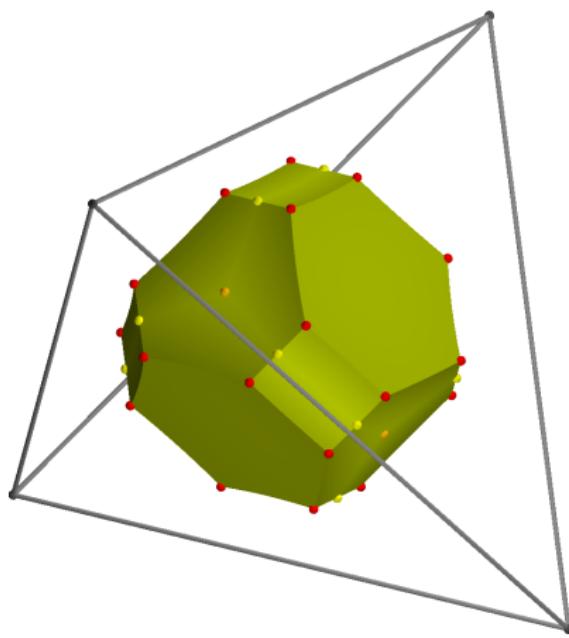
maximizes the volume within $\mathcal{S}^d(1)$.

Furthermore: the maximizer is unique for $d \geq 4$.

Example: $d = 3$ (Keep in mind: $s_1 = 2, s_2 = 3, s_3 = 7$)

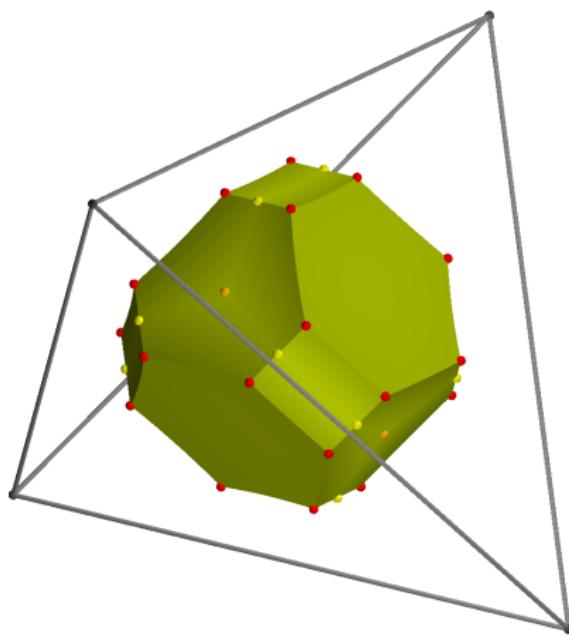


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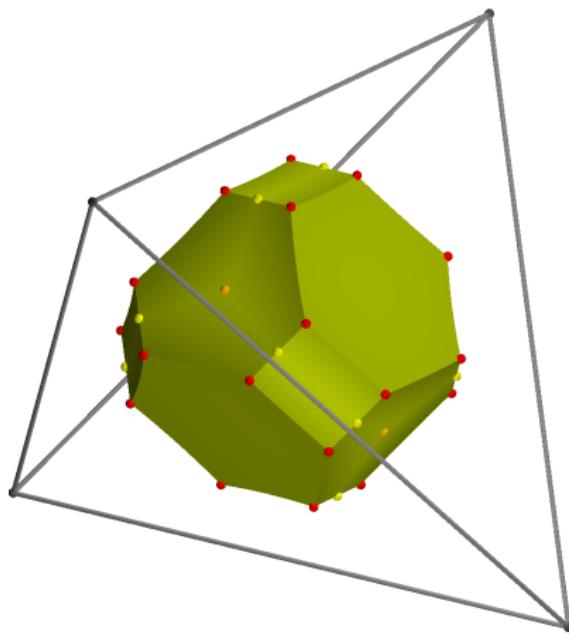
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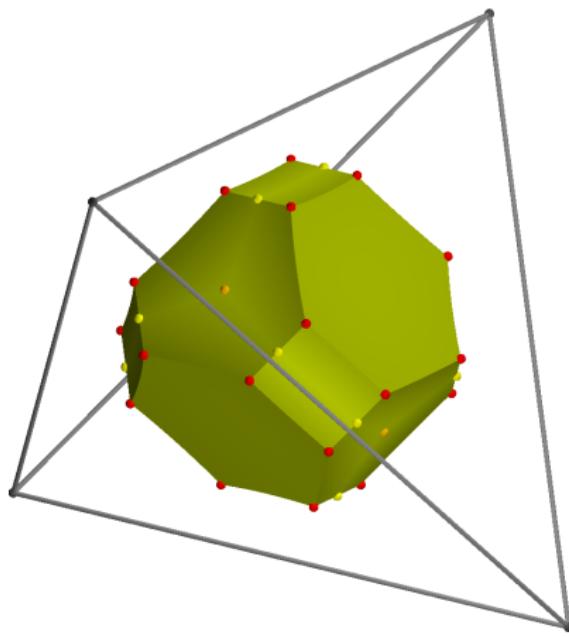
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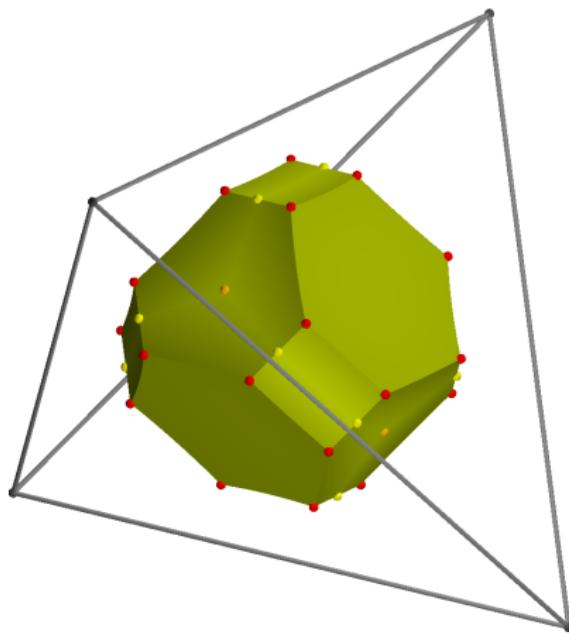
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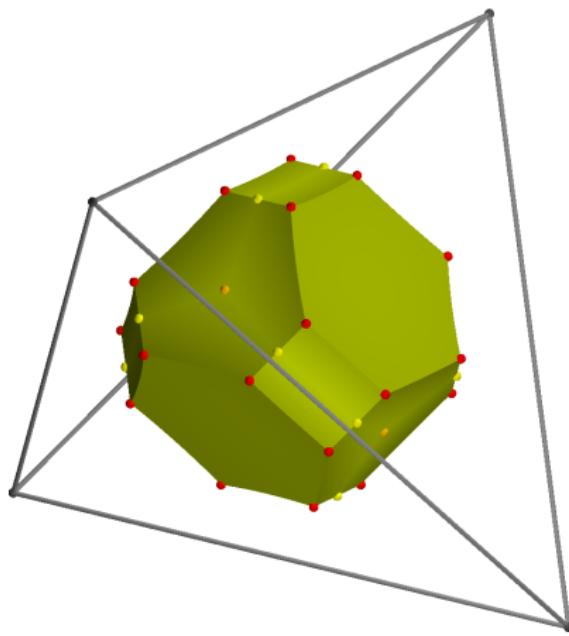
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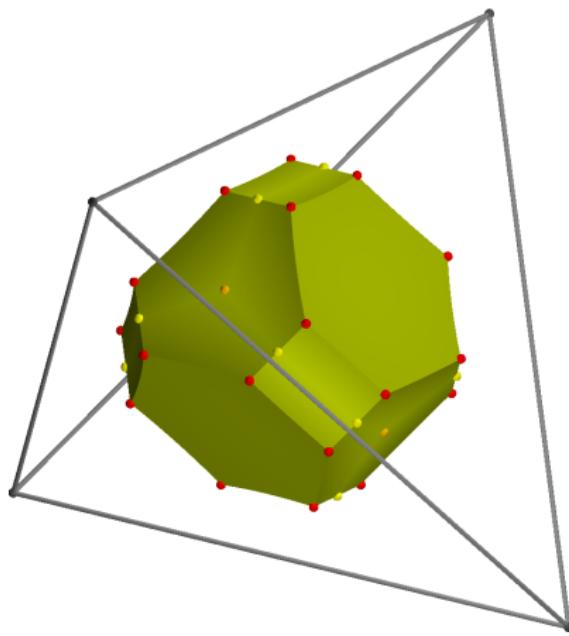
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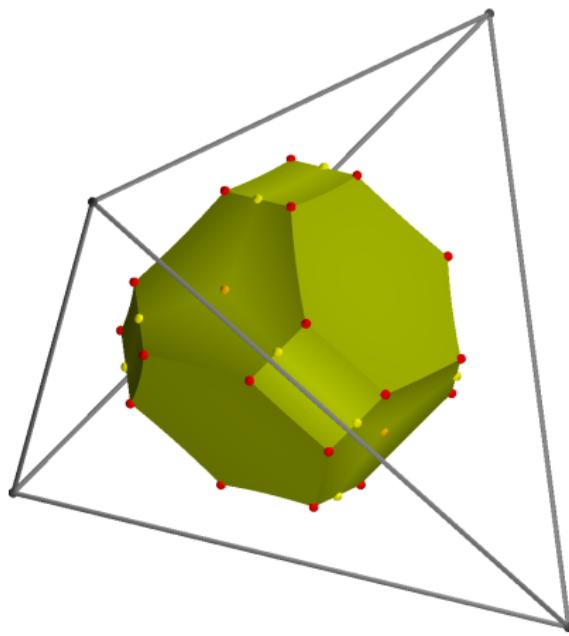
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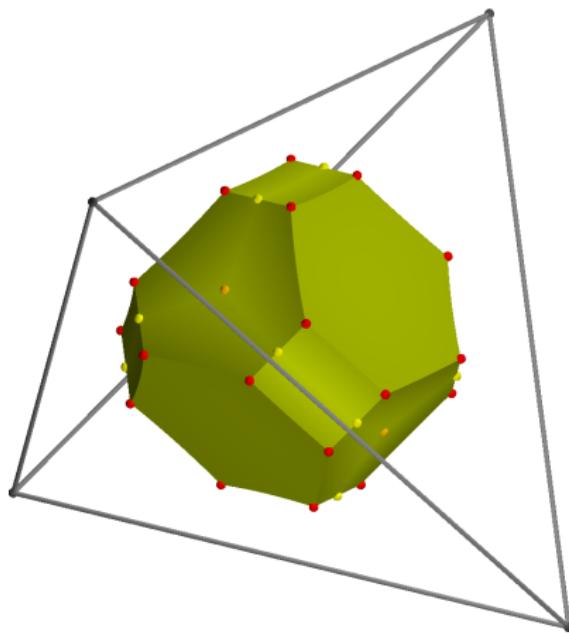
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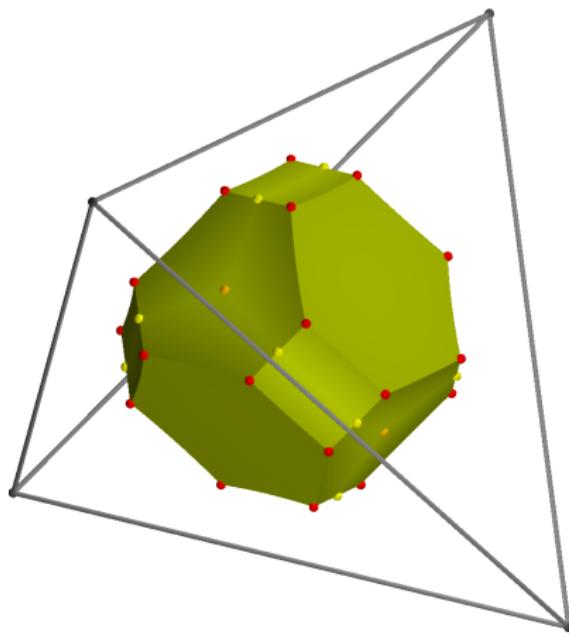
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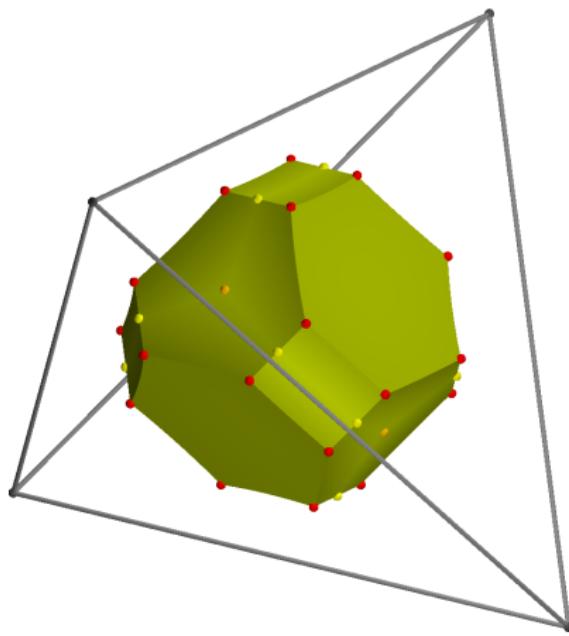
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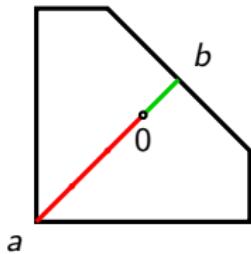
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Coefficient of asymmetry

For a compact convex set $K \subseteq \mathbb{R}^d$ with $0 \in \text{int}(K)$, the coefficient of asymmetry with respect to 0 is

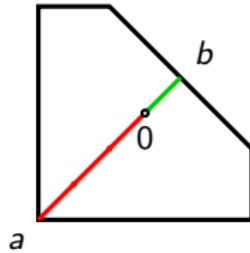
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Observe

One has $\text{ac}(K, 0) \geq 1$ and

$$\text{ac}(K, 0) = 1 \iff K \text{ is 0-symmetric}$$

Theorem (Mahler)

For every compact convex set $K \subseteq \mathbb{R}^d$ one has:

$$\text{int}(K) \cap \mathbb{Z}^d = \{0\} \quad \Rightarrow \quad \text{vol}(K) \leq (1 + \text{ac}(K, 0))^d.$$

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Upper bounds on $\text{ac}(P, 0)$ imply upper bounds on $\text{vol}(P)$ for $P \in \mathcal{P}^d(1)$.

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- The maximum asymmetry coefficient is $s_{d+1} - 2$.

Observe

- For $S \in \mathcal{S}^d(1)$ with $\text{int}(S) \cap \mathbb{Z}^d = \{0\}$ one has

$$\text{ac}(S, 0) = \frac{1}{\beta_{d+1}} - 1,$$

where β_{d+1} is the smallest barycentric coordinate of 0.

- $\text{conv}(s_1 e_1, \dots, s_d e_d)$ – unique minimizer of β_{d+1} within $\mathcal{S}^d(1)$
- $\Rightarrow \text{conv}(s_1 e_1, \dots, s_d e_d)$ – unique maximizer of the asymmetry coefficient within $\mathcal{S}^d(1)$.
- The maximum asymmetry coefficient is $s_{d+1} - 2$.

This can be extended

Theorem (A. & Krümpelmann & Nill 2015)

The simplex $\text{conv}(0, s_1 e_1, \dots, s_d e_d)$ is the unique polytope in $\mathcal{P}^d(1)$ maximizing the coefficient of asymmetry.

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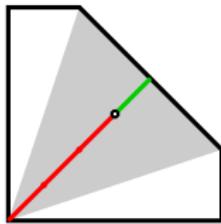
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Corollary (A. & Krümpelmann & Nill 2015)

$$\text{vol}(P) \leq (s_{d+1} - 1)^d \quad \forall P \in \mathcal{P}^d(1)$$

Proof idea

Consider an ‘extremal chord’ (starting in a vertex) of $P \in \mathcal{P}^d(1)$. There is a lattice simplex of dimension $\leq d$ which is at least as asymmetric as P (see figure).



Further results (1): A. & Krümpelmann & Nill 2015

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- Characterization of equality cases involves some number-theoretic considerations.

Further results (2): A. & Krümpelmann & Nill 2015

Sharp bounds on the lattice diameter for

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We re-use an argument from a paper by A. & Wagner & Weismantel 2011

A. & Krümpelmann & Nill (work in progress)

'Nearly optimal' volume bounds for $S \in \mathcal{S}^d(k)$ with $k \geq 1$.

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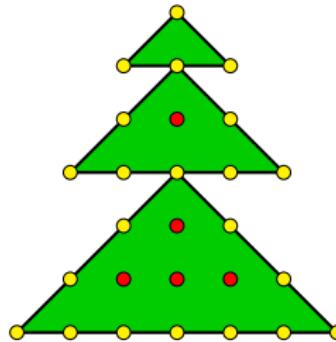
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Thank you for your attention!
Happy Christmas time!



1 Introduction

2 Lower bounds

3 Upper bounds on volume of simplices

4 Product-sum inequalities

5 Improvements

6 Further results

7 Outlook

8 Bibliography

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