Geometry of ν-Tamari lattices

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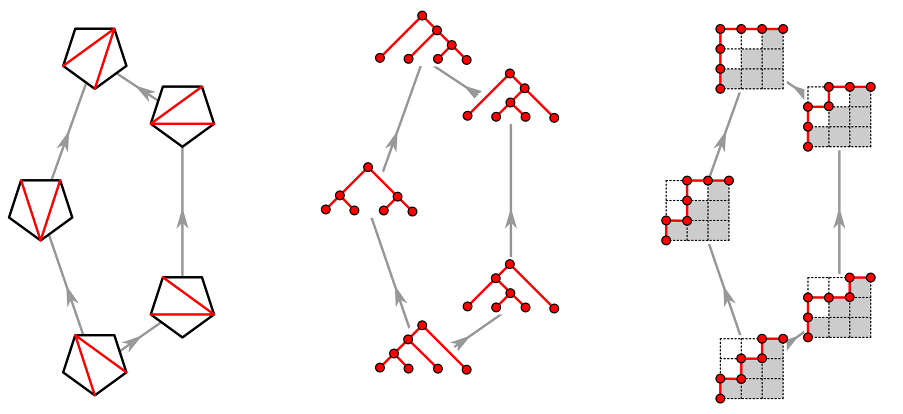
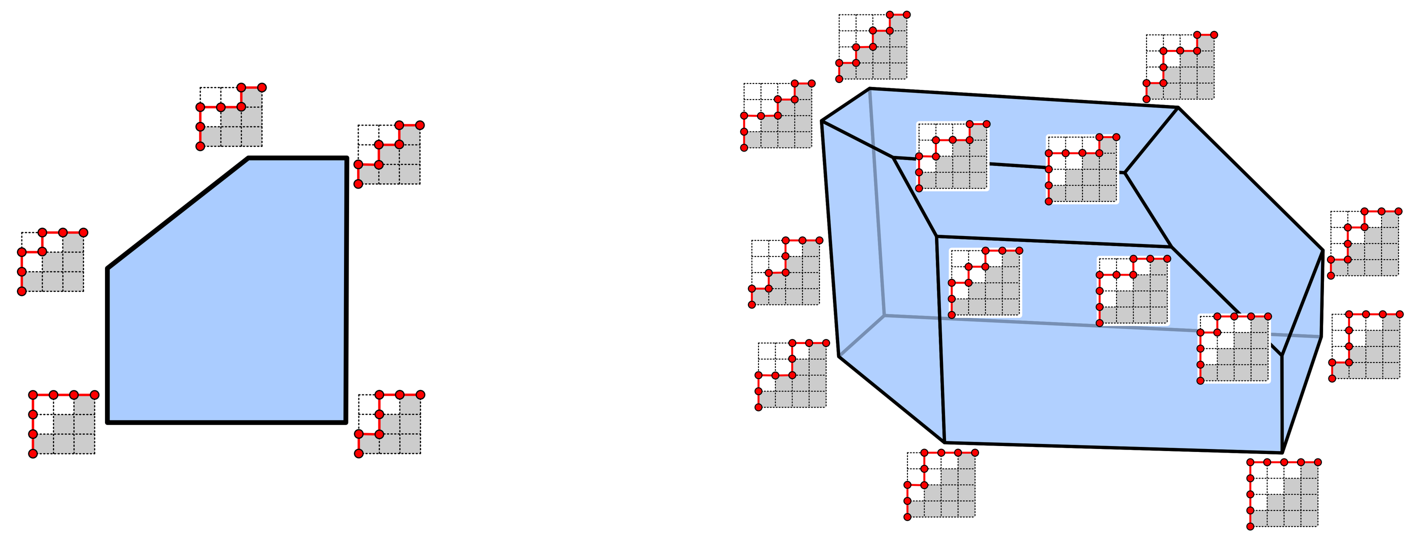
Geometry of the Tamari lattice

Tamari lattice: poset of Catalan objects

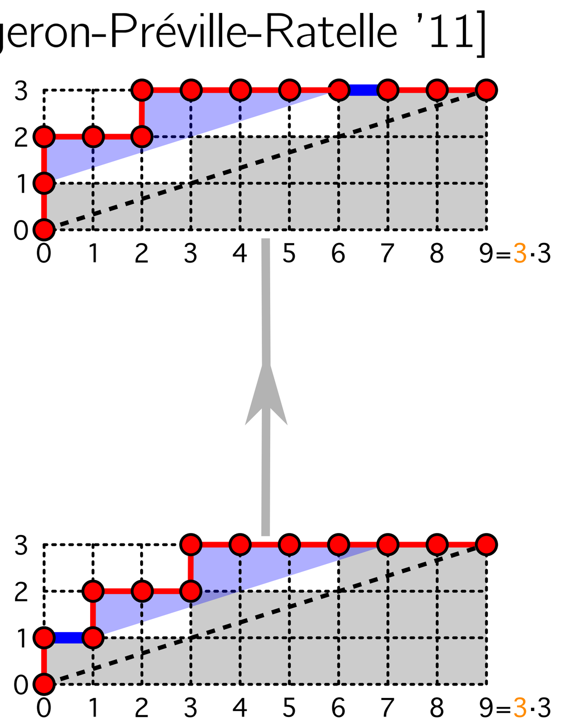


The Hasse diagram of the Tamari lattice is isomorphic to the edge graph of a simple polytope: the **associahedron** [Haiman '84, Lee '89, Gelfand-Kapranov-Zelevinsky '90, Fomin-Zelevinsky '03, Postnikov '09,...]

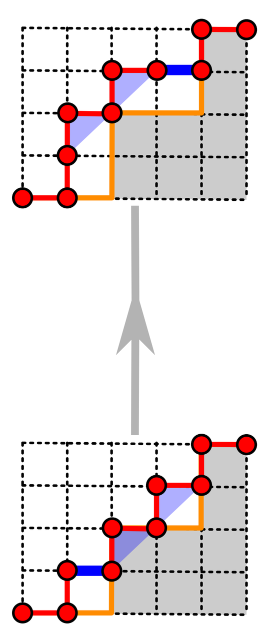


Extensions of the Tamari lattice

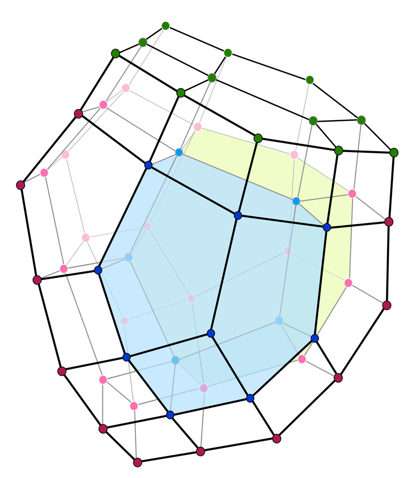
m-**Tamari lattice**: poset of Fuss-Catalan paths
[Bergeron-Préville-Ratelle '11]



ν-**Tamari lattice**: poset of lattice paths above ν
[Préville-Ratelle-Viennot '14]

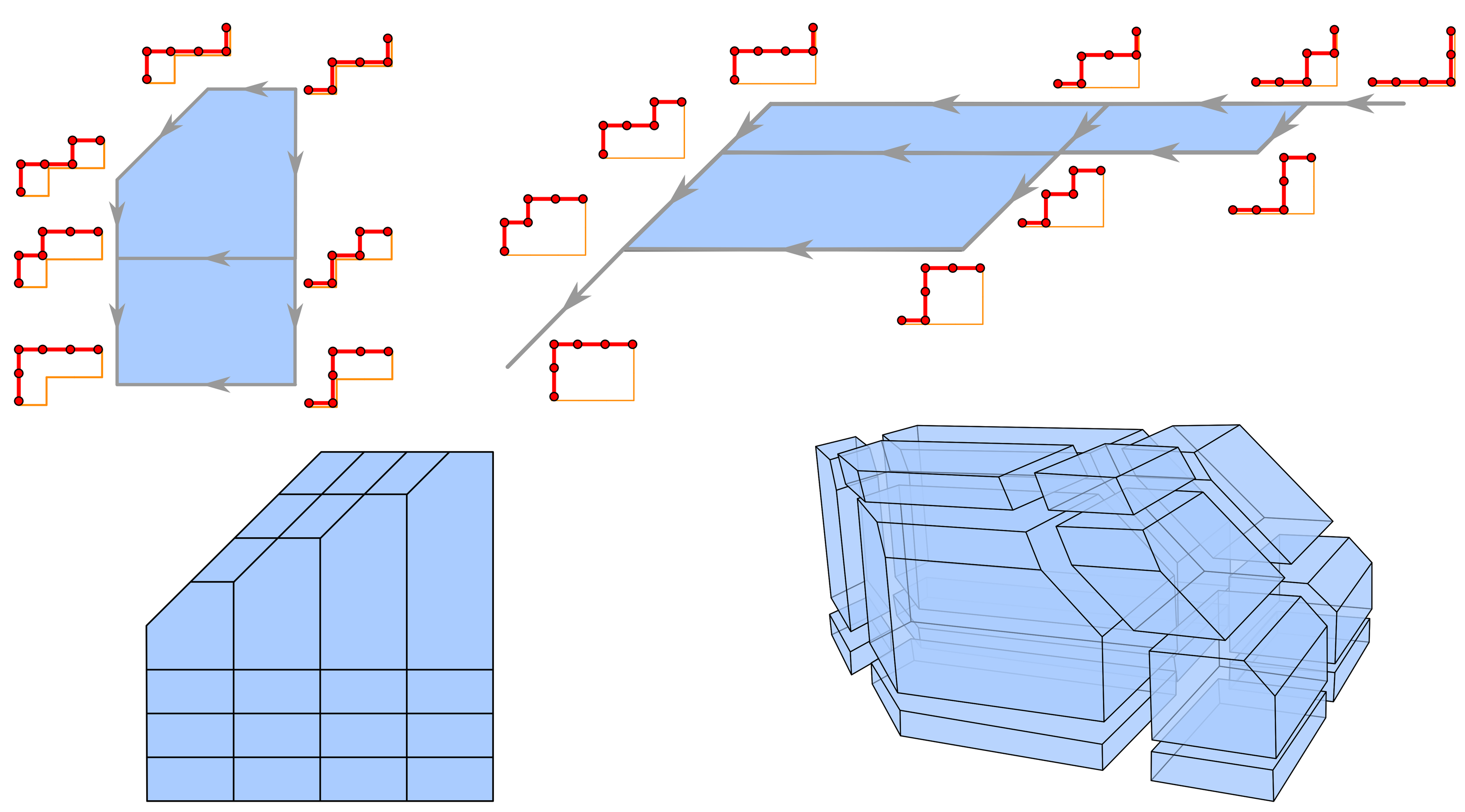


Are there geometric realizations of *m*-Tamari lattices analogous to the associahedron?
[Bergeron '12]



Theorem 1 [Ceballos-Padrol-S.]

The Hasse diagram of the ν-Tamari lattice is isomorphic to the edge graph of a polyhedral complex: the ν-**associahedron**. Its cells are cartesian products of associahedra.
In the Fuss-Catalan case, the ν-associahedron is a *tropical* subdivision of an associahedron.

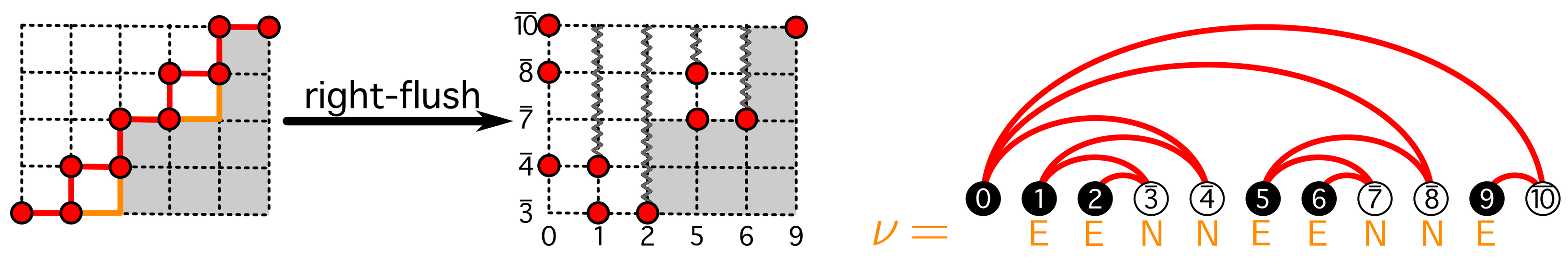


4-Tamari, $n = 3$ 2-Tamari, $n = 4$

Associahedral triangulations

Theorem 2 [Ceballos-Padrol-S.]

Lattice paths above ν are in bijection with $(I, \overline{J})$-**trees**



(here $I = \{0,1,2,5,6,9\}$, $\overline{J} = \{3,4,7,8,10\}$).
ν-Tamari covering relations $\iff$ flips of $(I, \overline{J})$-trees

Following [Gelfand-Graev-Postnikov '97], $(I, \overline{J})$-trees index the maximal simplices of a flag regular triangulation of a subpolytope $\mathcal{U}_{I,\overline{J}} \subset \Delta_I \times \Delta_{\overline{J}}$, where

$$\Delta_I \times \Delta_{\overline{J}} := \operatorname{conv}\left\{(\mathbf{e}_i, \mathbf{e}_{\overline{j}}) : \mathbf{e}_i \in I, \mathbf{e}_{\overline{j}} \in \overline{J}\right\} \subset \mathbb{R}^I \oplus \mathbb{R}^{\overline{J}}$$

$$\mathcal{U}_{I,\overline{J}} := \operatorname{conv}\left\{(\mathbf{e}_i, \mathbf{e}_{\overline{j}}) : \mathbf{e}_i \in I, \mathbf{e}_{\overline{j}} \in \overline{J}, i \leq j\right\}$$

We call it the $(I, \overline{J})$-**associahedral triangulation** $\mathfrak{A}_{I,\overline{J}}$
(for $I = [n], \overline{J} = [\overline{n}]$ it is dual to a $(n-1)$-associahedron.)

Proof of Theorem 1

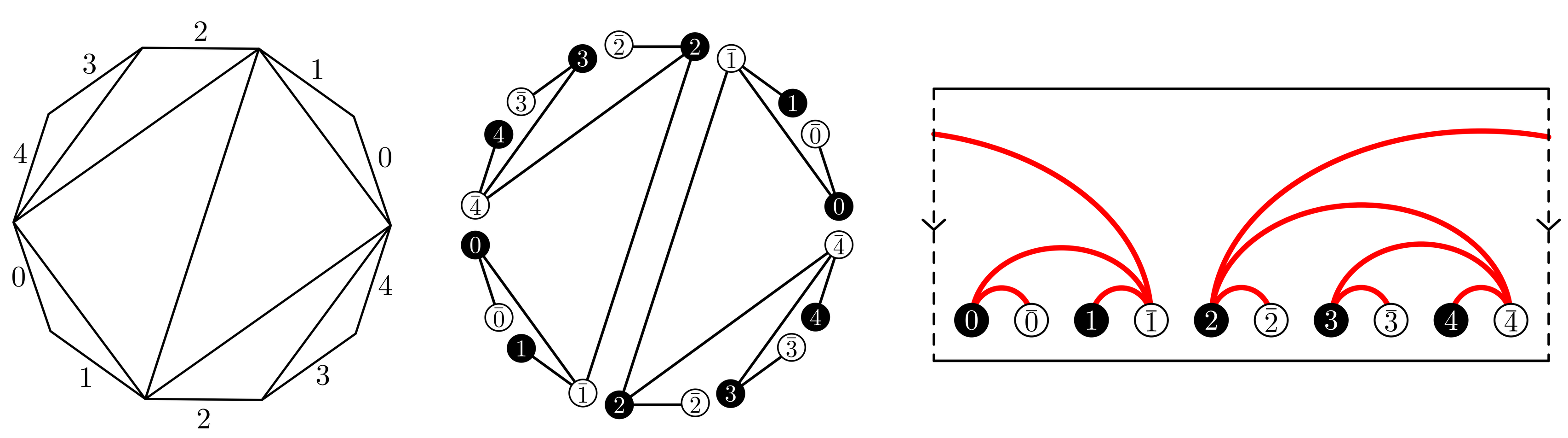
By [Develin-Sturmfels '04, Fink-Rincón '15], regular triangulations of subpolytopes of $\Delta_n \times \Delta_{\overline{m}}$ are in bijection with combinatorial types of arrangements of $n+1$ tropical hyperplanes in $\mathbb{TP}^m$. Apply this to the triangulation $\mathfrak{A}_{I,\overline{J}}$ to construct the ν-associahedron.

Theorem 3 [Ceballos-Padrol-S.]

The *h*-vector of $\mathfrak{A}_{I,\overline{J}}$ is given by the ν-**Narayana numbers**
$h_\ell = |\{\text{lattice paths above } \nu \text{ with exactly } \ell \text{ valleys}\}|$

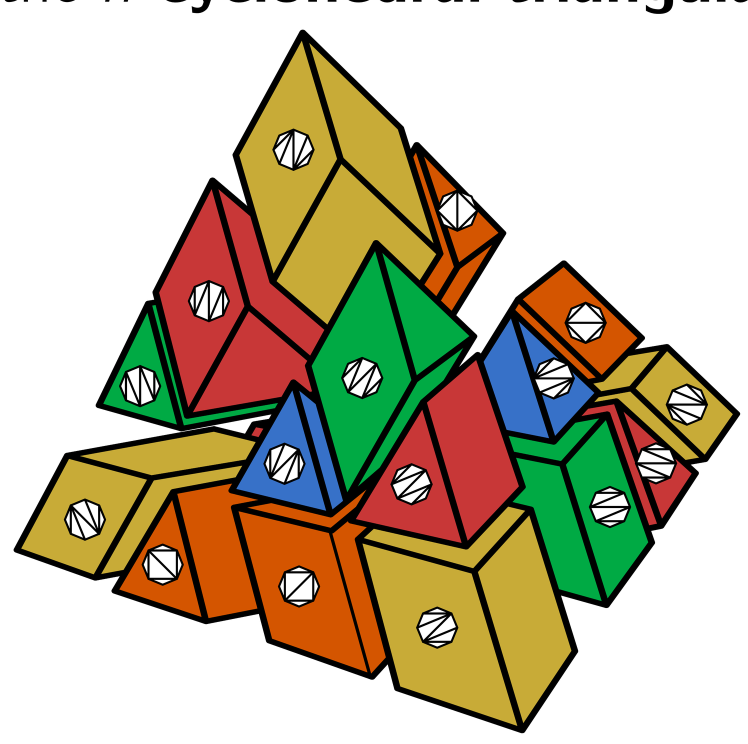
Type *B* extensions

Centrally symmetric triangulations of a $(2n+2)$-gon are in bijection with **cyclic** $([n], [\overline{n}])$-**trees**



Theorem 4 [Ceballos-Padrol-S.]

Cyclic $([n], [\overline{n}])$-trees index the maximal simplices of a flag regular Gorenstein triangulation of $\Delta_n \times \Delta_{\overline{n}}$, dual to a *n*-cyclohedron: the *n*-**cyclohedral triangulation** $\mathfrak{C}_n$.



Restricting $\mathfrak{C}_n$ to $\Delta_I \times \Delta_{\overline{J}}$ we get

- $(I, \overline{J})$-**cyclohedral triangulation**
- **simplicial** $(I, \overline{J})$-**cyclohedra**
- **simple** $(I, \overline{J})$-**cyclohedra**
- **type** *B* $(I, \overline{J})$-**Tamari poset** (not always a lattice ☹)