Rational Harnack Curves on Toric Surfaces

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Abstract

Harnack curves are a family of real algebraic curves who are distinguished because their topology is well understood, meaning that Hilbert's 16th problem is solved for these curves. Harnack curves have several characterizations in terms of their Amoebas, which are particularly nice. In this work we show how rational Harnack curves in any toric surface can be explicitly parametrized using Cox coordinates.

1. Hilbert's 16th problem

In 1876, Harnack proved that real projective curves of degree d have at most $\dfrac{d^2-3d+4}{2}$ connected components. More generally, if the curve $\mathbb{R}C$ consists of the real zeros of a real polynomial two variable polynomial f with newton polytope Δ, then $\mathbb{R}C$ has at most $g+1$ components inside the real toric variety $\mathbb{R}X_\Delta$, where g is the number of interior points of Δ and is called the *arithmetic genus*. Curves which attain this maximum of components are called *M-curves*.

The first part of Hilbert's 16th problem asks which are all the possible topological types of M-curves with respect to the coordinate axes of X_Δ. The problem is still open, however, it has been solved for Harnack curves.

2. Harnack curves

Let $l_1, \dots, l_n$ be the coordinate axes of $\mathbb{R}X_\Delta$ and let $d_1, \dots, d_n$ be the integer lengths of the sides of Δ corresponding to $l_1, \dots, l_n$ respectively.

Definition. A smooth real algebraic curve $\mathbb{R}C \subseteq \mathbb{R}X_\Delta$ is called a smooth *Harnack* curve if the following conditions hold:

- The number of connected components of $\mathbb{R}C$ is $g+1$ where g is the arithmetic genus.
- Only one component O of $\mathbb{R}C$ intersects $l_1 \cup \cdots \cup l_n$.
- O can be divided in n disjoint arcs, $\alpha_1 \dots \alpha_n$, cyclically ordered, such that $\alpha_j \cap l_j$ consists in d_j points (counted with multiplicities) and $\alpha_j \cap l_k = \emptyset$ for $j \neq k$.

These curves were first defined by Mikhalkin who solved Hilbert's 16 problem for them.

Theorem. *[Mik00] If $\mathbb{R}C$ is a smooth Harnack curve, the topology of $(\mathbb{R}X_\Delta, \{l_1 \dots, l_n\}, \mathbb{R}C)$ is unique up to Δ.*

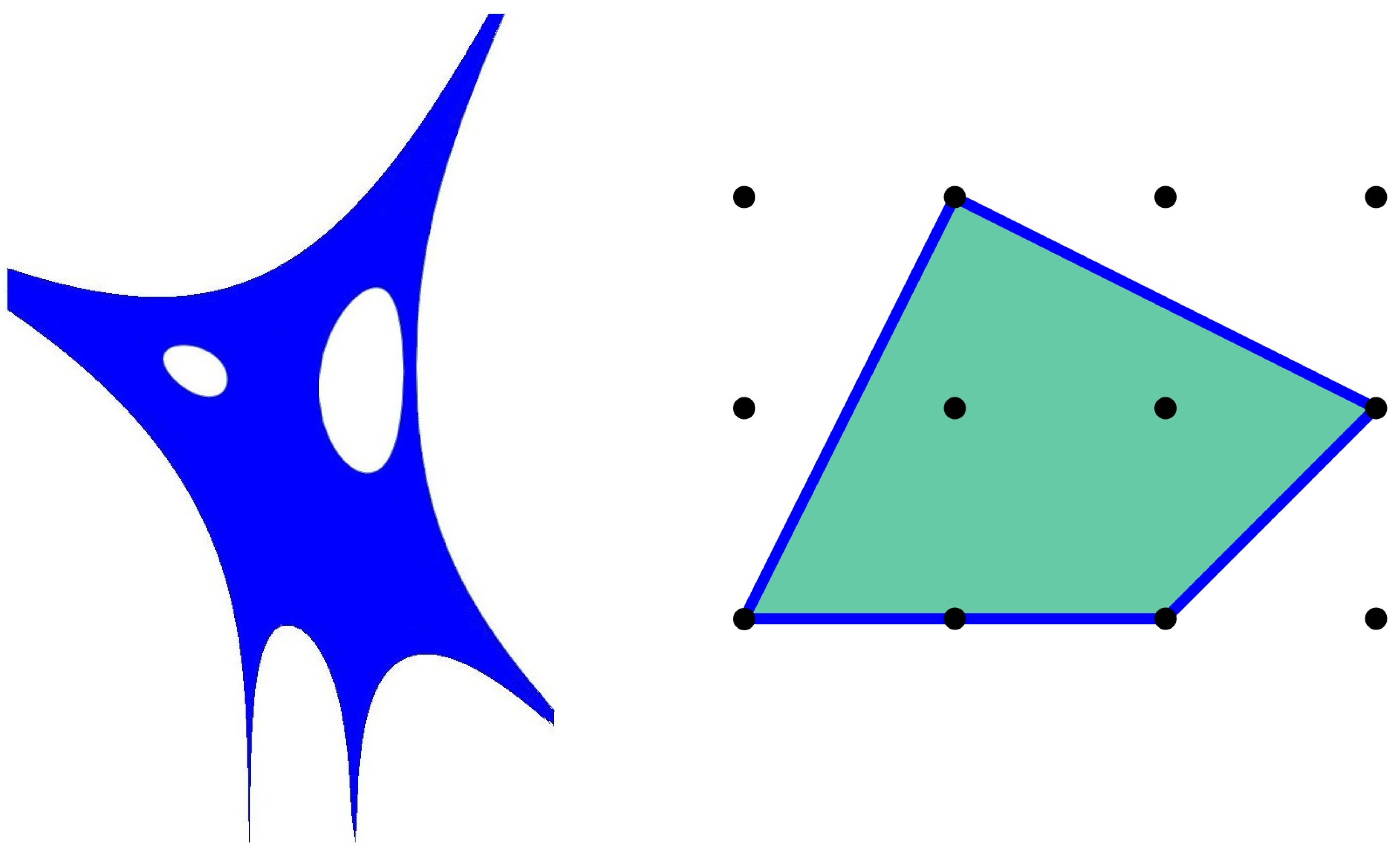
Harnack curves may also be defined for singular curves. A singular real curve $\mathbb{R}C$ is a Harnack curve if it is the limit of smooth Harnack curves where some of the components which do not intersect the coordinate axes contract to points.

3. The amoeba of a Harnack curve

Let $\mathrm{Log} : (\mathbb{C}^*)^2 \to \mathbb{R}^2$ be the map

$$\mathrm{Log}(z,w) := (\log|z|, \log|w|)$$

Definition. The *amoeba* of a polynomial f is $\mathcal{A} := \{\mathrm{Log}(z,w) \mid (z,w) \in (\mathbb{C}^*)^2 \;\; f(z,w) = 0\}$. In other words, it is the image of $C \cap (\mathbb{C}^*)^2$ under Log.



The amoeba and the Newton polytope of a Harnack curve given by

$$f(x,y) = xy^3 + x^3y + 10x^2y + 10xy - x^2 + 4x - 2.$$

Amoebas are a powerful tool for studying algebraic varieties and have strong connections to tropical geometry. The name amoeba comes from its tentacles which extend to infinity orthogonal to the side of Δ. The connected components of the complement of the amoeba are convex and each has a different point integer point in Δ associated to it.

Theorem. *[Mik00, MR01, PR10] The following are equivalent:*

1. *$\mathbb{R}C$ is Harnack curve*
2. *The map $\mathrm{Log}|_{C\cap(\mathbb{C}^*)^2}$ has only real singularities.*
3. *The map $\mathrm{Log}|_{C\cap(\mathbb{C}^*)^2}$ is at most 2-to-1.*
4. *The amoeba $\mathcal{A}$ has maximal area, that is $\pi^2\,\mathrm{area}(\Delta)$*
5. *The real part of the amoeba, $\mathcal{A}_\mathbb{R} = \mathrm{Log}(C \cap (\mathbb{R}^*)^2)$, has maximal total curvature, that is $2\pi\,\mathrm{area}(\Delta)$.*

4. Parametrization of rational Harnack curves

A rational curve C is a curve with geometric genus 0. This means that there is a rational map from $\phi : \mathbb{CP}^1 \to C$. It can be done using the Cox homogeneous coordinates of X_Δ:

$$\phi([r:s]) = \left[a_{1,0} \prod_{m=1}^{d_1} (r - a_{1,m}s) : \cdots : a_{n,0} \prod_{m=1}^{d_n} (r - a_{n,m}s) \right]_{X_\Delta} \qquad (1)$$

We call the constants $\{a_i, j\}_{i,j}$ the parameters. If C is a real curve, then all parameters must be real or in real conjugate pairs.

Definition. A rational real curve $\mathbb{R}C$ is *cyclical* if there exists a parametrization as in (1) such that if $1 \le k < m \le n$ then for all $1 \le i \le d_k$ and $1 \le j \le d_m$ we have that $a_{k,i} < a_{m,j}$.

If $\mathbb{R}C$ is a rational Harnack curve, then it by definition it must be a cyclical curve and for this case we have that the special component is $O = \phi(\mathbb{RP}^1)$. For a cyclical curve to be a Harnack curve, it must have the g isolated double points that come from contracted ovals without any other singularities. Our main result is that this is always the case.

Theorem. *Let $\mathbb{R}C$ be rational real curve. The following conditions are equivalent:*

- *$\mathbb{R}C$ is a Harnack curve.*
- *$\mathbb{R}C$ is cyclical.*

The proof of the theorem consists in considering each cyclical parametrization as a point in $\mathbb{R}^N$ with its parameters as coordinates. The space of cyclical parametrizations is connected. So by showing that the set of parametrizations corresponding to Harnack curves is non empty, closed and open subspace, we have that they must be the same. We use the following

Lemma. *The limit of Harnack curves remains Harnack, even if the Newton polytope degrades to a smaller polytope.*

Taking a Harnack curve with Newton polytope sufficiently large we can degrade it to a curve with smaller Newton polytope by making the roots of two adjacent sides equal, therefore 'cutting' the corresponding vertex. By the lemma the curve remains Harnack.

References

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