Enumerating lattice 3-polytopes

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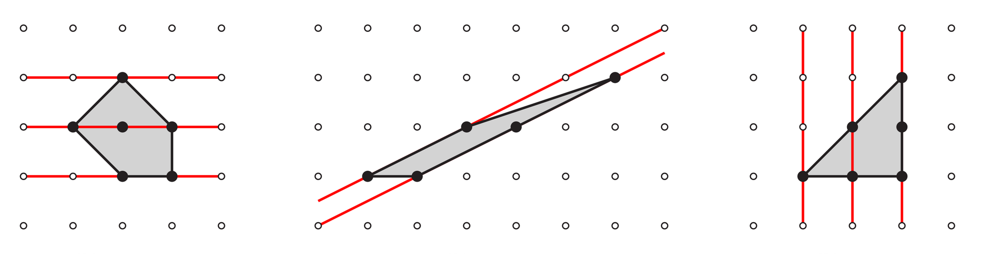
Basic definitions

Lattice d**-polytope** P	=	convex hull of a finite set of points in $\mathbb{Z}^d$ with $\mathrm{aff}(P) = \mathbb{R}^d$
Size of P	=	the number of lattice points in P: $\lvert P \cap \mathbb{Z}^d \rvert$
Width of P **with respect to the linear functional** $f : \mathbb{R}^d \to \mathbb{R}$	=	length of the interval $f(P)$
Width of P	=	the minimum $\mathrm{width}(P, f)$ among all non-constant linear functionals f

The width of P can also be interpreted as the minimum lattice distance between two lattice hyperplanes enclosing P



Equivalence

A **unimodular transformation** is a linear integer map $t : \mathbb{R}^d \to \mathbb{R}^d$ that preserves the lattice. That is,

$$t(x) = A \cdot x + b, \; x \in \mathbb{R}^d$$

for $A \in \mathbb{Z}^{d \times d}$, $\det(A) = \pm 1$ and $b \in \mathbb{Z}^d$. Two lattice d-polytopes P and Q are **equivalent** if there exists a unimodular transformation t such that $t(P) = Q$.

Size and width are invariant under unimodular transformations.

We consider "equivalence classes of" lattice d-polytopes

Consider, for each $n \geq d+1$, the following set:

$$\mathcal{P}_d(n) := \{ \text{(classes of) lattice } d\text{-polytopes of size } n \}$$

It is known that

- $\lvert \mathcal{P}_1(n) \rvert = 1$ (a segment of lenght $n-1$), for all $n \geq 2$.
- $\lvert \mathcal{P}_2(n) \rvert < \infty$ for all $n \geq 3$ (Pick's formula).
- $\lvert \mathcal{P}_d(n) \rvert = \infty$ for all $d \geq 3$ and $n \geq d+1$ (for example, Reeve tetrahedra for $d=3$, $n=4$).

GOAL (and course of action)

We want to classify all lattice 3-polytopes (via their size n)

- **Classification of $\mathcal{P}_3(4)$ (empty tetrahedra) (White 1964, [6])**

$$\mathcal{P}_3(4) = \{ T(p,q),\; 0 \leq p \leq q,\; \gcd(p,q) = 1 \}$$

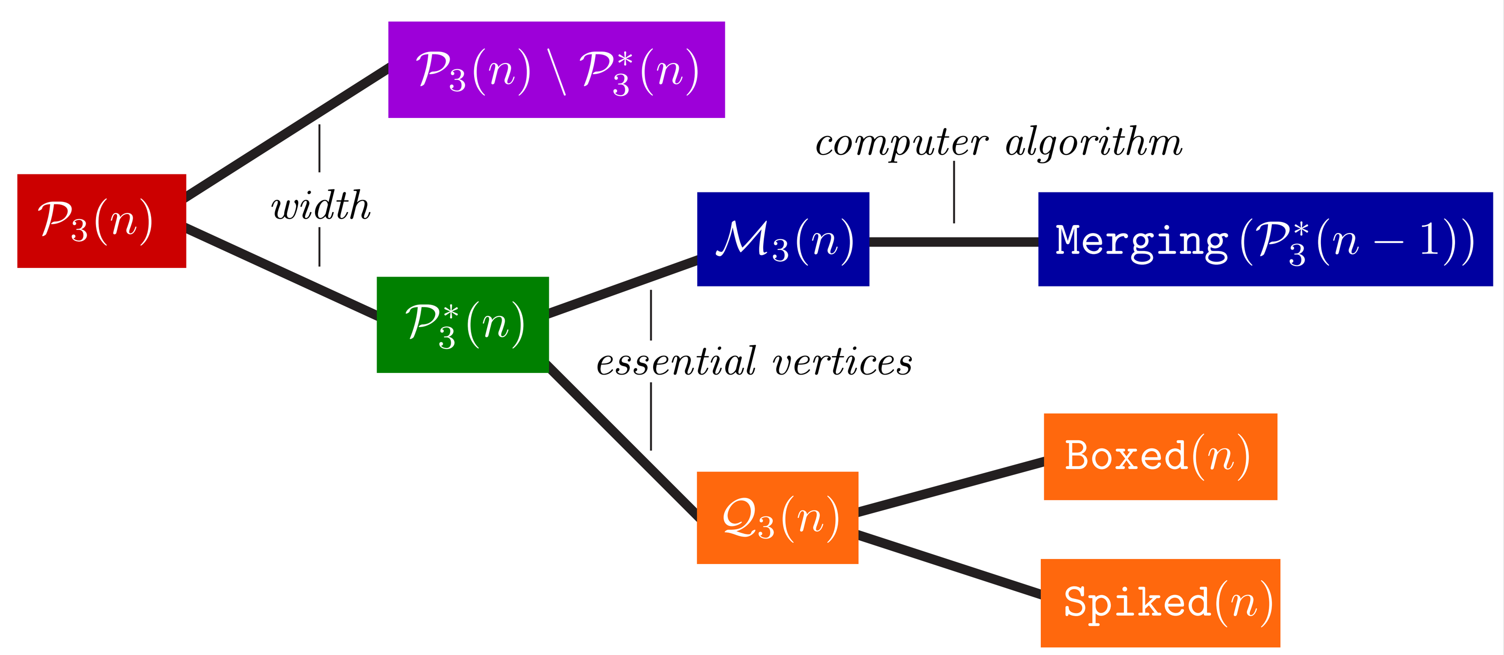
where $T(p,q) := \mathrm{conv}\,((0,0,0),(1,0,0),(0,0,1),(p,q,1))$, of width one. Moreover, $T(p,q)$ is equivalent to $T(p',q)$ if and only if $p' = \pm p^{\pm 1} \pmod{q}$.

- **Classification of $\mathcal{P}_3(5)$ and $\mathcal{P}_3(6)$ (Blanco-Santos, 2014-15, [2, 3])** (classification done via oriented matroids)

Size	4	5	6
width 1	∞	∞	∞
width 2	0	9	74
width 3		0	2

- For all $n \geq 7$...

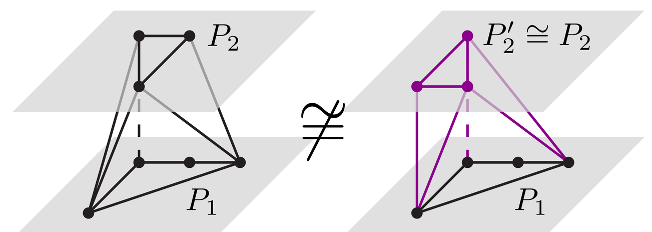
...follow the colors!!!!



Lattice d-polytopes of width 1

A lattice d-polytope P of size n and width one consists of the convex hull of two lattice polytopes P_1 and P_2, of dimensions $d_1, d_2 \leq d-1$, and of sizes $n_1, n_2 \geq 1$, $n_1 + n_2 = n$, lying in consecutive parallel lattice hyperplanes.

- In $d=1$, the only polytope of width one is the segment of length one (size $n=2$).
- In $d=2$, each polytope of width one is determined by the number of lattice points n_1 and n_2 in each of the parallel lines.
- For $d \geq 3$, there are infinitely many ways of positioning (rotating) one with respect to the other. That is, **for $d \geq 3$ and $n \geq d+1$, there exist infinitely many lattice d-polyotpes of size n and width one**.



References

[1] M. Blanco, C. Haase, J. Hofmann, F. Santos, The finiteness threshold width of lattice polytopes. Preprint, September 2016, arXiv:1607.00798.

[2] M. Blanco and F. Santos. Lattice 3-polytopes with few lattice points. *SIAM J. Discrete Math.* 30(2) (2016), 669–686.

[3] M. Blanco and F. Santos. Lattice 3-polytopes with six lattice points. *SIAM J. Discrete Math.* 30(2) (2016), 687–717.

[4] M. Blanco and F. Santos. On the enumeration of lattice 3-polytopes. Preprint, January 2016, arXiv:1601.02577.

[5] C. Haase, G. M. Ziegler. On the Maximal Width of Empty Lattice Simplices. *Eur. J. Comb.* 21 (2000), no. 1, 111–119.

[6] G. K. White. Lattice tetrahedra. *Canadian J. Math.* 16 (1964), 389–396.

Dimension 4

In dimension 4, the main ingredient used in dimension 3 (the fact that $\lvert \mathcal{P}_3^*(n) \rvert < \infty$), fails:

THEOREM: (Haase-Ziegler, 2000, [5]) There exist infinitely many lattice emtpy 4-simplices (elements of $\mathcal{P}_4(5)$) of width 2.

THEOREM: (Blanco-Haase-Hofmann-Santos, 2015, [1])

$$\lvert \{ P \in \mathcal{P}_4(n) \mid \mathrm{width}(P) > 2 \} \rvert < \infty, \quad \text{for all } n \geq 5$$

Lattice polytopes of width > 1

$$\mathcal{P}_d^*(n) := \{ P \in \mathcal{P}_d(n) \mid \mathrm{width}(P) > 1 \}$$

THEOREM: (Blanco-Santos, 2014, [2]) For each $n \geq 4$, $\lvert \mathcal{P}_3^*(n) \rvert < \infty$.

So there are only finitely many lattice 3-polytopes of width larger than one of each size.

Enumeration results (on 3-polytopes of width > 1)

size	width 2	width 3	width 4	total	approx. comp. time
5	9	0	0	9	[2]
6	74	2	0	76	[3]
7	477	19	0	496	14 min.
8	2524	151	0	2675	70 min.
9	10862	836	0	11698	7 hours
10	40885	4148	2	45035	48 hours
11	137803	18635	26	156464	20 days

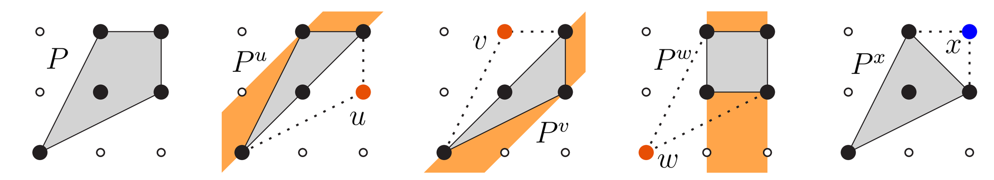
Software: MATLAB

Quasiminimal vs. Merged polytopes

Let $P \in \mathcal{P}_d^*(n)$, and let $v \in \mathrm{vert}(P)$. We denote $P^v := \mathrm{conv}(P \setminus \{v\} \cap \mathbb{Z}^d) \subset \mathbb{R}^d$. This polytope has size $n-1$ but it is not necessarily d-dimensional.
We say that v is an **essential** vertex if P^v has width zero ($(d-1)$-dimensional), or one.
We say that v is **NOT essential** if P^v has width > 1, in which case, $P^v \in \mathcal{P}_d^*(n-1)$.

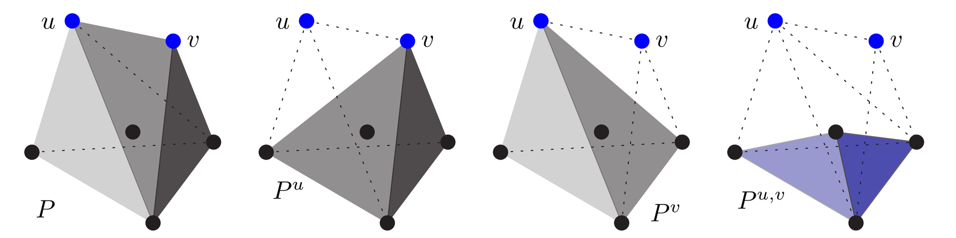
- We say that a polytope $P \in \mathcal{P}_d^*(n)$ is **quasiminimal** if all but at most one of its vertices are **essential** (P^v has width 0 or 1, for all but at most one of its vertices v).

$$\mathcal{Q}_d(n) := \{ P \in \mathcal{P}_d^*(n) \mid P \text{ is quasiminimal} \}$$



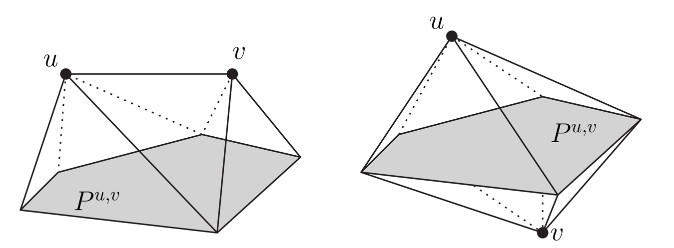
- We say that a polytope $P \in \mathcal{P}_d^*(n)$ is **merged** if at least two of its vertices u, v are **NOT essential** (such that P^u, P^v have width > 1) AND the polytope $P^{u,v} := \mathrm{conv}(P^u \cap P^v \cap \mathbb{Z}^d) = \mathrm{conv}(P \setminus \{u,v\} \cap \mathbb{Z}^d)$ is still d-dimensional.

$$\mathcal{M}_d(n) := \{ P \in \mathcal{P}_d^*(n) \mid P \text{ is merged} \}$$



These two sets are disjoint for every d and every $n \geq d+1$. But, *do they cover all $\mathcal{P}_d^*(n)$?*

EXCEPTIONS: If a polytope $P \in \mathcal{P}_d^*(n)$ has at least two **NOT essential** vertices, and for all pairs of **NOT essential** vertices u and v (that is, with $P^u, P^v \in \mathcal{P}_d^*(n-1)$), the polytope $P^{u,v}$ is $(d-1)$-dimensional, then $P \notin \mathcal{Q}_d(n)$ and $P \notin \mathcal{M}_d(n)$.



Dimension 3: Notice that $\mathcal{P}_3^*(4) = \emptyset$ and $\mathcal{P}_3^*(5) = \mathcal{Q}_3(5)$.

THEOREM: There is a single lattice 3-polytope of width larger than one that is neither *quasiminimal* nor *merged*, and it is of size $n = 6$. That is,

$$\lvert \mathcal{P}_3^*(6) \setminus (\mathcal{Q}_3(6) \cup \mathcal{M}_3(6)) \rvert = 1 \quad \text{and} \quad \mathcal{P}_3^*(n) = \mathcal{Q}_3(n) \sqcup \mathcal{M}_3(n), \; \forall n \geq 7.$$

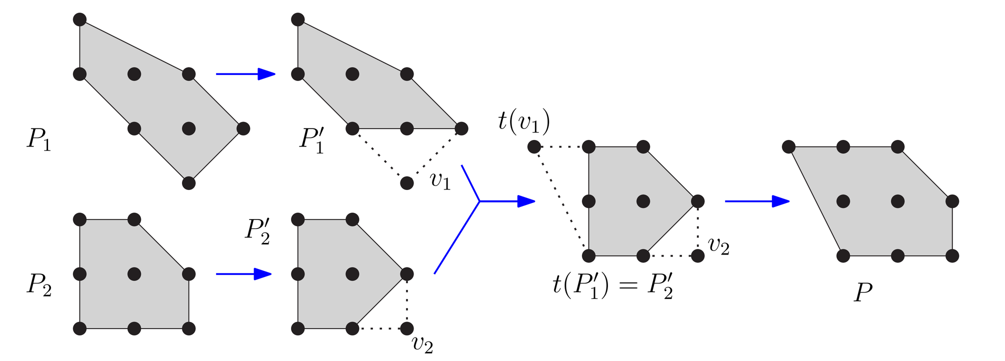
Merged polytopes

ALGORITHM: Merging

INPUT: some finite list L of lattice d-polytopes of size $n-1$ and width > 1.
OUTPUT: the list $L' = \mathtt{Merging}(L)$ of all lattice d-polytopes P of size n and width greater than one, containing subpolytopes $P_1, P_2 \in L$ and such that $\mathrm{conv}(P_1 \cap P_2 \cap \mathbb{Z}^d)$ is d-dimensional and of size $n-2$.

For each two polytopes $P_1, P_2 \in L$, and for each vertex v_1 of P_1 and v_2 of P_2:

1. Let $P_1' = \mathrm{conv}(\mathbb{Z}^d \cap P_1 \setminus \{v_1\})$ and $P_2' = \mathrm{conv}(\mathbb{Z}^d \cap P_2 \setminus \{v_2\})$.
2. Check if P_1' and P_2' are d-dimensional and unimodularly equivalent. If they are, let $t : \mathbb{Z}^d \to \mathbb{Z}^d$ be an equivalence sending P_1' to P_2'. (t may be not unique, but there are finitely many possibilities for it; do step 3 for each).
3. If the size of $P := \mathrm{conv}(t(P_1) \cup P_2)$ equals n, add P to the output list L'.

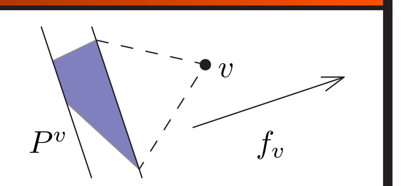


Dimension 3: By definition of $\mathcal{M}_3(n)$, and since $\mathcal{P}_3^*(n-1)$ is a finite list:

$$\mathcal{M}_3(n) = \mathtt{Merging}(\mathcal{P}_3^*(n-1)), \text{ for all } n \geq 7.$$

Quasiminimal polytopes

Let $P \in \mathcal{Q}_d(n)$, for every essential vertex $v \in \mathrm{vert}(P)$, let $f_v : \mathbb{R}^d \to \mathbb{R}$ be an integer linear functional that gives width one (or zero) to P^v.

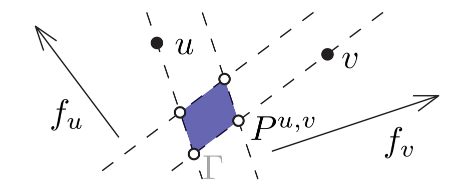


We distinguish 2 cases:

- If the set $\{f_v : v \text{ is essential vertex of } P\}$ linearly spans $(\mathbb{R}^d)^*$, then we can find d linearly independent f_v. We call these polytopes **boxed**, because

most of their lattice points lie in the vertices of a d-parallelepiped Γ

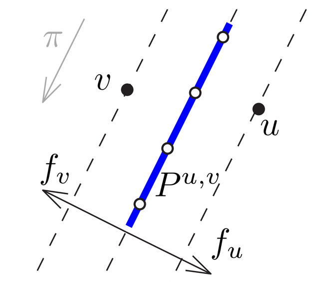
$$\mathtt{Boxed}_d(n) := \{ P \in \mathcal{Q}_d(n) \mid P \text{ is boxed} \}$$



- If the set $\{f_v : v \text{ is essential vertex of } P\}$ does not linearly span $(\mathbb{R}^d)^*$, then there is a projection π that respects all f_v. We call these polytopes **spiked**, because

most of their lattice points lie in a segment

$$\mathtt{Spiked}_d(n) := \{ P \in \mathcal{Q}_d(n) \mid P \text{ is spiked} \}$$



Clearly, $\mathcal{Q}_d(n) = \mathtt{Boxed}_d(n) \cup \mathtt{Spiked}_d(n)$, $\forall n, d$.

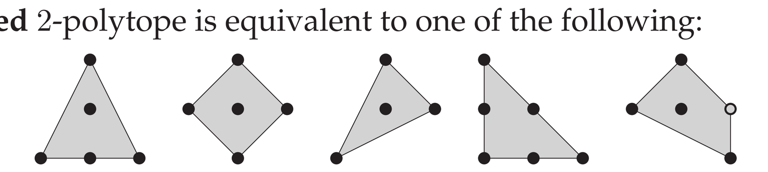
BOXED POLYTOPES:

The lattice points of a boxed d-polytope are d essential vertices plus some of the 2^d vertices of this parallelepiped. Hence a boxed d-polytope has size $\leq d + 2^d$.

In dimension 2, boxed polytopes have size ≤ 6 and their classification is done via exhaustive search among the polytopes of those sizes (of which there are finitely many).

LEMMA: Every **boxed** 2-polytope is equivalent to one of the following:

$$\bigcup_{n=3}^{6} \mathtt{Boxed}_2(n) =$$



The grey dot represents the non-essential vertex.

In dimension 3, boxed polytopes have size ≤ 11. Because $\lvert \mathcal{P}_3^*(n) \rvert < \infty$ for all $n \geq 4$, the list of boxed 3-polytopes is finite: $\lvert \bigcup_{n=4}^{11} \mathtt{Boxed}_3(n) \rvert < \infty$. Moreover, the parallelepiped Γ can only be the unit cube:

LEMMA: Let $P \in \mathtt{Boxed}_3(n)$, for $n \geq 7$, then P consist of 3 essential vertices plus some of the vertices of the unit cube $[0,1]^3$.

Boxed 3-polytopes are enumerated by theoretically bounding the possibilities for the 3 vertices outside of $[0,1]^3$ and then trying every possibility via computer search.

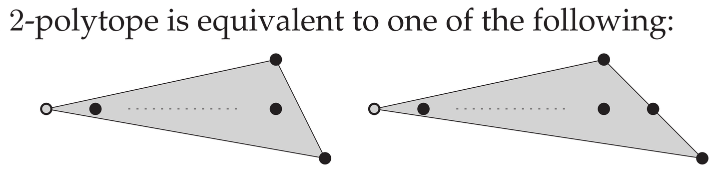
SPIKED POLYTOPES:

The lattice projection of a spiked polytope via π (a projection that preserves all the f_v) has one dimension lower and properties very similar to those of a quasiminimal polytope.

In dimension 2, spiked polytopes project to the unique quasiminimal 1-polytope, the segment of length 2:

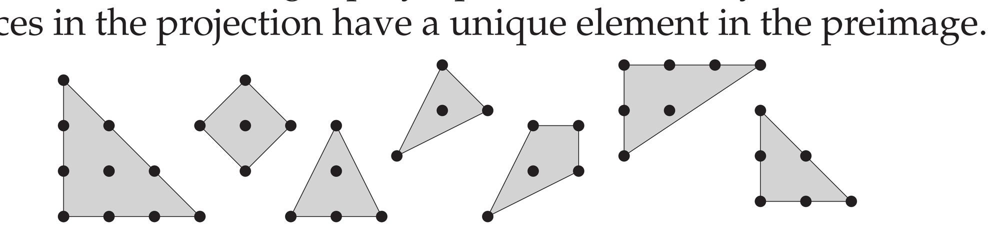
LEMMA: Every **spiked** 2-polytope is equivalent to one of the following:

$$\bigcup_{n \geq 3} \mathtt{Spiked}_2(n) =$$



In dimension 3, the list of possible projections of a spiked polytope is still finite:

THEOREM: Every polytope $P \in \mathtt{Spiked}_3(n)$, for $n \geq 7$, projects to one of the following 2-polytopes in such a way that all of the vertices in the projection have a unique element in the preimage.



Spiked 3-polytopes are explicitly described for each size $n \geq 7$: each of those polygons have finitely many polytopes projecting to them with the necessary properties and size n.