**Enumerating lattice 3-polytopes**

**GOAL (and course of action)**

We want to classify all lattice 3-polytopes (via their size $n$):

- **Classification of $P(1)$ (empty tetrahedron)**
  \[ P(1) = \{ T(\pi, 0), 0 \leq \pi \leq 2\pi, \pi(x,y,z) = 1 \} \]
  where $T(\pi, z) = \cos(0.01z)$, $(0.01z)$, $(0.01z)$, $(0.01y + 1)$, of width of $1$.
  Moreover, $T(\pi, y)$ is equivalent to $T(\pi, x)$ if and only if $\pi = \pi + 2\pi$ (mod 2).

- **Classification of $P(2)$ and $P(3)$**

**Basic definitions**

Lattice $d$-polytopes $P$ are convex hull of a finite set of points in $\mathbb{Z}^d$ with all $(P)$ $\mathbb{R}^d$.

- **Size of $P$**
  The number of lattice points in $P$.

- **Width of $P$ with respect to the linear function $f$**
  $|f(P)|$.

- **Width of $P$**
  The minimum width of $P$ among all non-constant linear functions $f$.

The width of $P$ can also be interpreted as the minimum lattice distance between two lattice hyperplanes enclosing $P$.

**Lattice $d$-polytopes of width 1**

A lattice $d$-polytope $P$ of size $n$ and width one consists of the convex hull of two lattice polytopes $P_1$ and $P_2$ of dimensions $d_1, d_2, \ldots, d_k$, and of size $n_1, n_2, \ldots, n_k$, lying in consecutive parallel lattice hyperplanes.

- **In $d = 3$**, the only polytope of width one is the segment of length one (size $n = 2$).
- **In $d > 3$**, each polytope of width one is determined by the number of lattice points $v_1, v_2, \ldots, v_k$ in each of the parallel lines.

For $d = 3$, there are infinitely many ways of positioning (rotating) one with respect to the other. That is, for $d = 3$, there exist infinitely many lattice $d$-polytopes of size $n$ and width one.

**EQUIVALENCE**

A unimodular transformation is a linear map $f: \mathbb{Z}^d \to \mathbb{Z}^d$ that preserves the lattice. That is, $f(x) = x + y$ for some $y \in \mathbb{Z}^d$.

For $A$ 2D $P(0,0)$, if $x$ and $y \in \mathbb{Z}^2$, two lattice polytopes $P$ and $Q$ are equivalent if there exists a unimodular transformation $f$ such that $f(P) = Q$.

Size and width are invariant under unimodular transformations.

We consider “equivalence classes” of lattice $d$-polytopes.

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