On the maximum dual volume of a canonical Fano polytope

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Background and main result

We consider a d-dimensional **canonical Fano polytope** P, i.e. a lattice polytope with exactly **one interior lattice point** that we suppose to be the origin.

By Hensley [4] the volume $\mathrm{vol}(P)$ is bounded just in terms of the dimension, but the known bounds are far from being sharp. The largest (in terms of volume) known examples of canonical Fano polytopes (described in [6]) are the following:

Largest known canonical Fano simplices $\quad R_{(d)} := S_{(d)} - \sum_{i=1}^{d} e_i, \qquad$ where $S_{(d)} := \mathrm{conv}\{\mathbf{0}, 2(s_d-1)e_d, s_{d-1}e_{d-1}, \ldots, s_1 e_1\}$

where $s_1 = 2,\ s_2 = 3,\ s_3 = 7,\ s_4 = 43,\ \ldots\ s_{i+1} := s_1 \cdots s_i + 1$ for $i \in \mathbb{Z}_{\geq 1}$ is the **Sylvester sequence**.

CONJECTURE: Sharp upper bound for the volume of P

The volume of $R_{(d)}$ is maximal among all the canonical Fano polytopes of dimension $d \geq 4$:

$$\mathrm{vol}(P) \leq \frac{1}{d!}\, 2(s_d-1)^2 \qquad \text{with equality if and only if } P = R_{(d)}$$

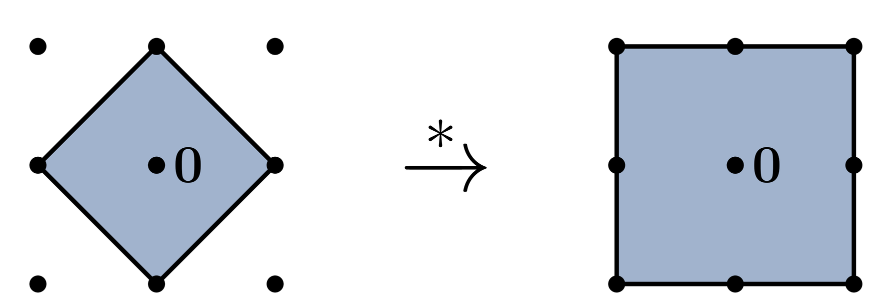
In small dimensions this problem has been solved via classifications (see [5]) and needs a separate formulation. For simplices the conjecture has been proven in [2].

The dual volume

We focus on the volume of the **dual** P^* of P, i.e

$$P^* := \{y \in (\mathbb{R}^d)^* : \langle y, x\rangle \geq -1 \text{ for every } x \in P\}$$

In general P^* is not a lattice polytope. If it is, we call P is a **reflexive polytope**.



MAIN THEOREM: Sharp upper bound for the volume of P^*

For $d \geq 4 \quad \mathrm{vol}(P^*) \leq \dfrac{1}{d!}\, 2(s_d-1)^2 \quad$ with equality if and only if $P^* = R_{(d)}$.

COROLLARY

If P is a reflexive polytope then

$$\mathrm{vol}(P) \leq \frac{1}{d!}\, 2(s_d-1)^2$$

with equality if and only if $P = R_{(d)}$.

COROLLARY

If X is d-dimensional toric Fano variety with at worst canonical singularities then its anti-canonical degree $(-K_X)^d$ is bounded by

$$(-K_X)^d \leq 2(s_d-1)^2$$

Overview of the proof

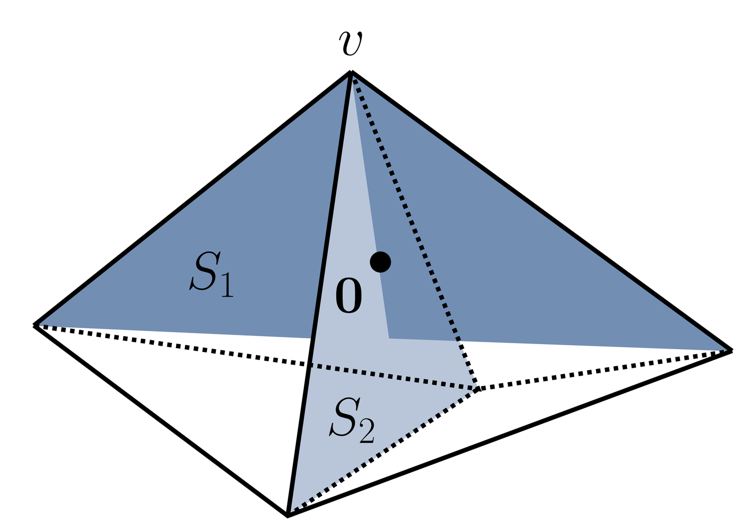
1. Minimal canonical Fano polytopes

Dualizing reverses inclusions:

$$P \subset Q \Rightarrow Q^* \subset P^*.$$

Hence we focus on **minimal** canonical Fano polytopes. These admit a **decomposition** in lower dimensional canonical Fano simplices.

$$P = \mathrm{conv}(S_1 \cup \cdots \cup S_t)$$



In the figure P decomposes into two canonical Fano simplices S_1 and S_2 sharing a common vertex v.

2. Bounds via monotonicity

A consequence of the decomposition of P in simplices is

$$P^* \subseteq S_1^* \times \cdots \times S_t^*.$$

We use the **monotonicity of the normalised volume** $\mathrm{Vol}(P) := d!\,\mathrm{vol}(P)$:

$$P^* \subseteq S_1^* \times \cdots \times S_t^* \ \Rightarrow\ \mathrm{Vol}(P^*) \leq \mathrm{Vol}(S_1^* \times \cdots \times S_t^*),$$

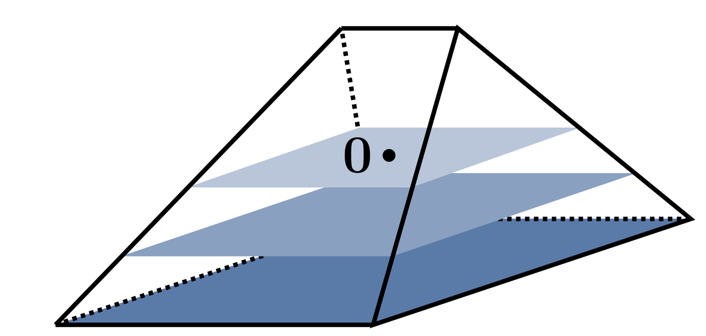
and the sharp bound for the volume of the dual of canonical Fano simplices [2]:

$$\begin{aligned}\mathrm{Vol}(P^*) &\leq \mathrm{Vol}(S_1^* \times \cdots \times S_t^*)\\ &= (d_1 + \cdots + d_t)! \prod_{i=1}^{t} \mathrm{vol}(S_i^*)\\ &\leq (d_1 + \cdots + d_t)! \prod_{i=1}^{t} \frac{1}{d!}\, 2(s_{d_i}-1)^2\end{aligned}$$

This allows us to prove most of the cases. We are left with the case in which the minimal polytope P decomposes in two $(d-1)$-dimensional simplices and a few cases in dimension up to 5.

3. A refined method via integration

Another consequence of the decomposition is a description of P^* as **union of slices**.



We can describe each slice in terms of a product of slices of the S_i^*. By using bounds for the volume of slices of simplices [1], we express (via an integration) the volume of P^* in terms of the barycentric coordinates of the simplices.

4. Remaining cases

We are left with a few cases in dimension up to 5. We solve them via **computer-assisted proofs**.

- If P decomposes in two lower dimensional simplices, we use the formula we obtained via integration together with some classification of barycentric coordinates in dimensions up to 4.
- If P decomposes in three lower dimensional simplices we use the decomposition to build the possible canonical minimal Fano polytopes starting from minimal canonical Fano simplices.

References

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[3] Gabriele Balletti, Alexander M. Kasprzyk, and Benjamin Nill *On the maximum dual volume of a canonical Fano polytope*, preprint, arXiv:1611.02455 [math.CO], 2016.

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[5] Alexander M. Kasprzyk, *Canonical toric Fano threefolds*, Canad. J. Math. **62** (2010), no. 6, 1293–1309.

[6] J. Zaks, M. A. Perles, and J. M. Wills, *On lattice polytopes having interior lattice points*, Elem. Math. **37** (1982), no. 2, 44–46.