Near-rational subdivisions of non-rational polytopes

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Basics

A *polytope* is a bounded set of the form $\{x \in \mathbb{R}^n : Ax \leq b\}$ with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.

A *polyhedral complex* is a finite collection of n-dimensional polytopes, called *cells* such that the intersection of any two polytopes is a face of both.

A *subdivision* of a polytope P is a polyhedral complex whose union of cells equals P.

A *triangulation* is a subdivision where all cells are simplices.

Rational polytopes

A *rational polytope* is a polytope of the form $\{x \in \mathbb{R}^n : Ax \leq b\}$ with $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$.

Alternatively, it is a polytope whose vertices have rational coordinates.

Rational polytopes have rational triangulations

Proposition

Every rational polytope P has a triangulation into rational simplices.

Proof.

Suffices to show that there is a triangulation into simplices whose vertices are vertices of P.

Let f be a function from the set of vertices of P to $\mathbb{R}$, and consider the convex hull of $\{(v, f(v)) : v \text{ is a vertex of } P\}$.

The lower hull of this polytope is a subdivision of P, and if f is generic it is a triangulation. $\square$

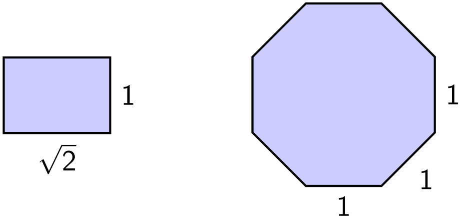
V-polytopes

Let $V \subset \mathbb{R}$ be a vector space over $\mathbb{Q}$.

A *V-polytope* is a polytope of the form $\{x \in \mathbb{R}^n : Ax \leq b\}$ with $A \in \mathbb{Q}^{m \times n}$, $b \in V^m$.

Alternatively, it is a polytope whose edge slopes (or facet normal slopes) are rational and whose vertices have coordinates in V.

Example: $V = \mathbb{Q}[\sqrt{2}]$



V-polytopes cannot be triangulated into V-simplices

Proposition

A V-polytope P can be triangulated into simplices which are V-polytopes if and only if P is a multiple of a rational polytope.

Proof sketch.

A simplex is a V-polytope if and only if it is a multiple of a rational polytope. $\square$

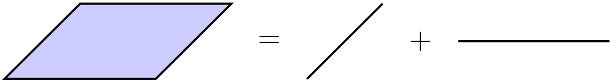
Products of simplices

Given polytopes $P_1, \ldots, P_r$, the *Minkowski sum* is defined to be

$$P_1 + \cdots + P_r := \{x_1 + \cdots + x_r : x_i \in P_i\}.$$

A *product of simplices* is a polytope P of the form $P = \Delta_1 + \cdots + \Delta_r$, where $\Delta_1, \ldots, \Delta_r$ are simplices and

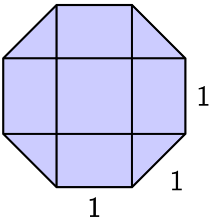
$$\dim(P) = \dim(\Delta_1) + \cdots + \dim(\Delta_r).$$



Main theorem

Theorem (Adiprasito, L., Pak, Temkin)

Every V-polytope can be subdivided into products of simplices, all of which are V-polytopes.



Motivation from log geometry

In toric geometry, toric varieties can be represented by fans of *toric monoids*, or *rational cones*.

Subdividing these rational fans, or equivalently, subdividing rational polyhedral complexes, corresponds to certain operations in toric geometry (resolution of singularities, semistable reduction...)

Motivation from log geometry (cont.)

In *log geometry*, one wants to consider fans of "R-toric monoids", where R is a valuation monoid. These monoids correspond to cones which are dual to the V-polytopes discussed above.

Because V-polytopes cannot be subdivided into V-simplices, we do not have the same strength of results as in the toric case. But using products of simplices gives us the best possible analogous result (polystable reduction).

Mixed subdivisions

Sturmfels ('94), Huber-Rambau-Santos ('00)

Let $P_1, \ldots, P_r$ be polytopes and $P := P_1 + \cdots + P_r$.

A *mixed subdivision* of P with respect to $P_1, \ldots, P_r$ is a subdivision of P where each cell C is given a label $(C_1, \ldots, C_r)$ such that the following hold:

1. For each i, C_i is a polytope whose vertices are vertices of P_i.
2. $C = C_1 + \cdots + C_r$.
3. If C, C' are two cells labeled $(C_1, \ldots, C_r)$ and $(C'_1, \ldots, C'_r)$, then for each i, $C_i \cap C'_i$ is a face of both C_i and C'_i.

Fine mixed subdivisions

A mixed subdivision is *fine* if for every cell C with label $(C_1, \ldots, C_r)$, each C_i is a simplex and

$$\dim(C) = \dim(C_1) + \cdots + \dim(C_r).$$

In particular, every cell of a fine mixed subdivision is a product of simplices.

Proposition

Let P_1, ..., P_r be polytopes and $P := P_1 + \cdots + P_r$. Then there exists a fine mixed subdivision of P with respect to P_1, ..., P_r.

Fine mixed subdivisions

Proposition

Let $P_1, \ldots, P_r$ be polytopes and $P := P_1 + \cdots + P_r$. Then there exists a fine mixed subdivision of P with respect to $P_1, \ldots, P_r$.

Proof.

For $1 \leq i \leq r$, let f_i be a function from the set of vertices of P_i to $\mathbb{R}$. Let $P_i^{f_i}$ be the convex hull of $\{(v, f(v)) : v \text{ is a vertex of } P_i\}$.

Consider the polytope $P_1^{f_1} + \cdots + P_r^{f_r}$. The lower hull of this polytope is a mixed subdivision of P. If the f_i are generic, it is a fine mixed subdivision. $\square$

Sketch of main proof

Theorem

Every V-polytope can be subdivided into products of simplices, all of which are V-polytopes.

Proposition (Main Proposition)

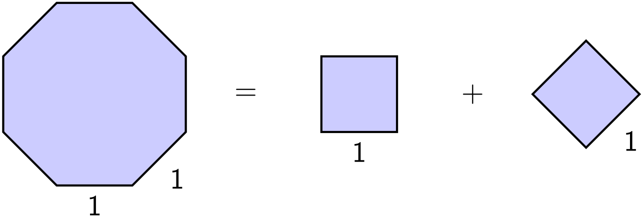
Every V-polytope can be written as $P_1 + \cdots + P_r$, where each P_i is a V-multiple of a rational polytope.

Proof of theorem.

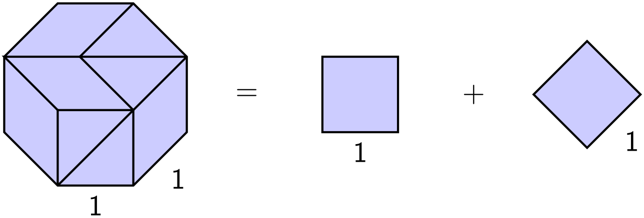
Write the V-polytope as $P = P_1 + \cdots + P_r$ as in the Proposition. Take a fine mixed subdivision of P with respect to $P_1, \ldots, P_r$.

Every cell C of the subdivision is a product of simplices with $C = C_1 + \cdots + C_r$, where each C_i is a V-simplex. Thus C is a V-polytope. $\square$

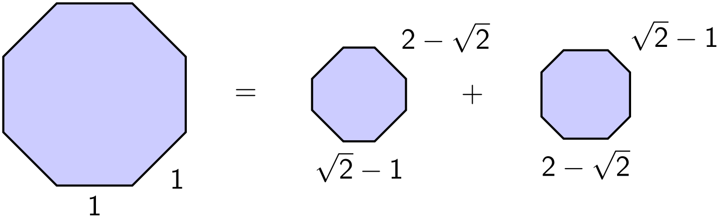
Example



Example



Example



Proof of main proposition

Proposition (Main Proposition)

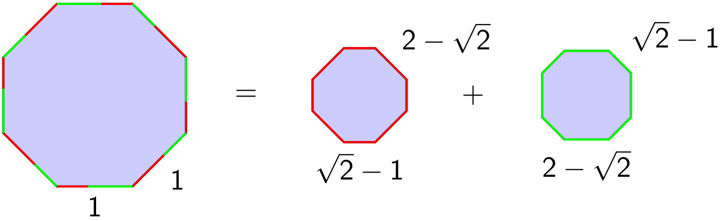
Every V-polytope P can be written as $P_1 + \cdots + P_r$, where each P_i is a multiple of a rational polytope.

Proof idea.

Choose elements $\beta_1, \ldots, \beta_r \in V$. For each edge e of P, write $e = e_1 + \cdots + e_r$, where each e_i is a β_i-multiple of a rational segment, and such that...

For each $1 \leq i \leq r$, construct a polytope P_i with the same 1-skeleton as P but with each edge e replaced with e_i. □

Example



$\beta_1 = \sqrt{2} - 1$ $\qquad$ $\beta_2 = 2 - \sqrt{2}$

Proof of main proposition

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Proof idea.

Choose elements $\beta_1, \ldots, \beta_r \in V$. For each edge e of P, write $e = e_1 + \cdots + e_r$, where each e_i is a β_i-multiple of a rational segment, and such that for every 2-face of P with edges $e, \ldots, f$, the edges $e_i, \ldots, f_i$ form a polygon.

For each $1 \leq i \leq r$, construct a polytope P_i with the same 1-skeleton as P but with each edge e replaced by e_i. $\square$

Proof of main proposition

Proposition

Let P be a polytope, and for each edge e of P, let e' be a segment parallel to e. Suppose that for every 2-face of P with edges $e, \ldots, f$, the edges $e', \ldots, f'$ form a polygon. Then there is a polytope with the same 1-skeleton as P but with each edge e replaced by e'.

Proof.

Reconstruct P' vertex-by-vertex from its graph. $\square$

Proof of main proposition (cont.)

Proof idea.

Choose elements $\beta_1, \ldots, \beta_r \in V$. For each edge e of P, write $e = e_1 + \cdots + e_r$, where each e_i is a β_i-multiple of a rational segment, and such that for every 2-face of P with edges $e, \ldots, f$, the edges $e_i, \ldots, f_i$ form a polygon.

For each $1 \leq i \leq r$, construct a polytope P_i with the same 1-skeleton as P but with each edge e replaced by e_i. □

To complete the proof, we need to show that we can always choose $\beta_1, \ldots, \beta_r \in V$ and decompositions $e = e_1 + \cdots + e_r$ as above.

In fact, this can be done so that $\beta_1, \ldots, \beta_r$ are all positive and form a basis for V over $\mathbb{Q}$!

Concluding remarks

Conclusion: Every V-polytope has a fine mixed subdivision with respect to $\dim_{\mathbb{Q}}(V)$ summands.

Result extends to polyhedral complexes of V-polytopes (which is what is needed on the algebraic geometry side).

Thank you!