Covering compact metric spaces greedily

Jan Rolfes
(University of Cologne)

Frank Vallentin
(University of Cologne)

Definitions and basic properties

Let (X, d) be compact metric space and $r \in \mathbb{R}_{>0}$. We define its *covering number* $\mathcal{N}(X, r)$ by

Definition 1

$$B(x, r) = \{y \in X : d(y, x) \leq r\}$$
$$\mathcal{N}(X, r) = \min\{|Y| : \ Y \subseteq X, \ \cup_{y \in Y} B(y, r) = X\}.$$

The covering number has e.g. applications in compressive sensing, approximation and probability theory, and machine learning.

We equip X with probability measure ω satisfying the following two conditions:

Properties of ω

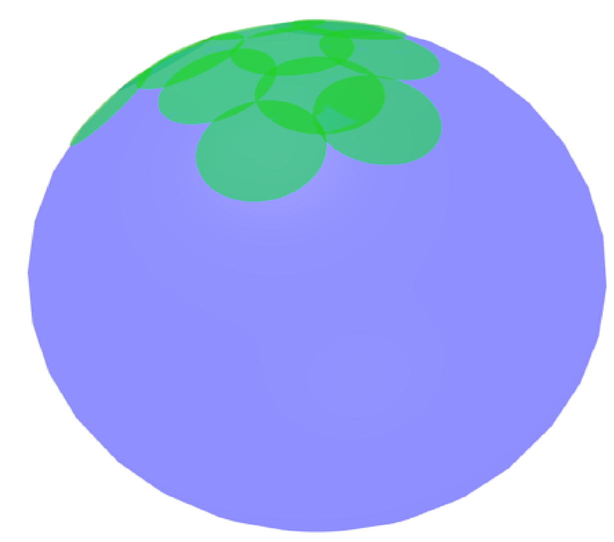
(a) $\omega(B(x, s)) = \omega(B(y, s))$ for all $x, y \in X$, and for all $s \geq 0$,

(b) $\omega(B(x, \varepsilon)) > 0$ for all $x \in X$, and for all $\varepsilon > 0$.

We denote $\omega(B(x, s))$ by ω_s.

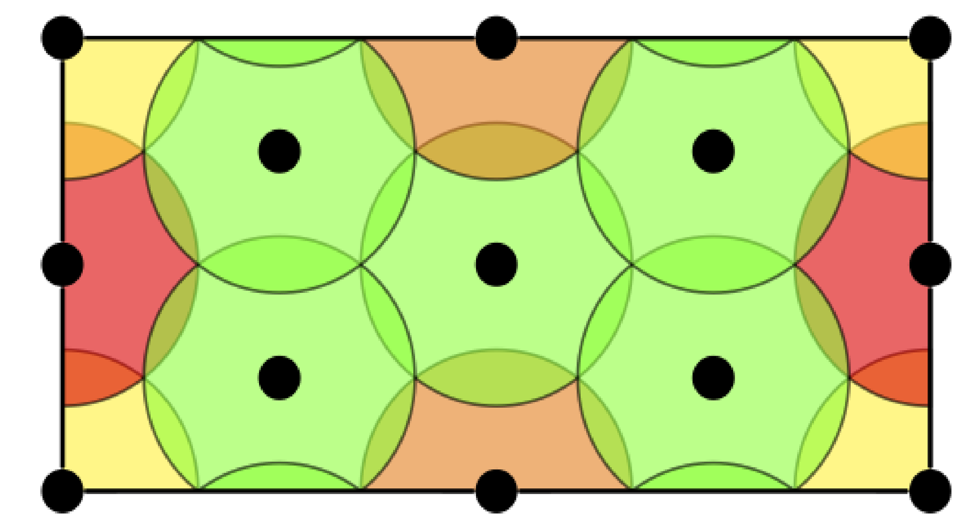
Examples

$X = S^2$



Green caps cover parts of a blue sphere.

$X = \mathbb{R}^2 / \left(\mathbb{Z}\begin{pmatrix}6.4\\0\end{pmatrix} + \mathbb{Z}\begin{pmatrix}0\\3.2\end{pmatrix}\right)$



Covering with 8 balls per torus:

- 5 inner green balls
- 1 yellow ball (corners are identified)
- 1 orange resp., red ball (edges are identified)

Greedy algorithm

Algorithm 1

1. $i \leftarrow 0$
2. $S_x^i = B(x, r - \varepsilon)$ for all $x \in X$
3. **while** $\bigcup_{j=1}^{i} B(y^j, r) \neq X$ **do**
4. $\quad i \leftarrow i + 1$
5. $\quad$ Choose $y \in X$ with $\omega(S_y^{i-1}) \geq \omega(S_x^{i-1})$ for all $x \in X$
6. $\quad y^i = y$
7. $\quad S_x^i = S_x^{i-1} \setminus S_y^{i-1}$ for all $x \in X$
8. **end while**

Analysis similar to Chvátal (1979) for weighted SET COVER:

- Consider infinite-dimensional LP-relaxation of $\mathcal{N}(X, r - \varepsilon)$
- Relaxation LP has optimal value $\frac{1}{\omega_{r-\varepsilon}}$
- Algorithm 1 defines: $g(x) = \begin{cases} \omega(S_{y^i}^{i-1})^{-1} & \text{if } x \in S_{y^i}^{i-1} \\ 0 & \text{otherwise.} \end{cases}$
- $f = \left(\ln\left(\frac{\omega_{r-\varepsilon}}{\omega_\varepsilon}\right) + 1\right)^{-1} g$ is feasible solution for the dual of LP
- f has objective value $\frac{|Y|}{\left(\ln\left(\frac{\omega_{r-\varepsilon}}{\omega_\varepsilon}\right)+1\right)} \leq \frac{1}{\omega_{r-\varepsilon}}$

Main Result

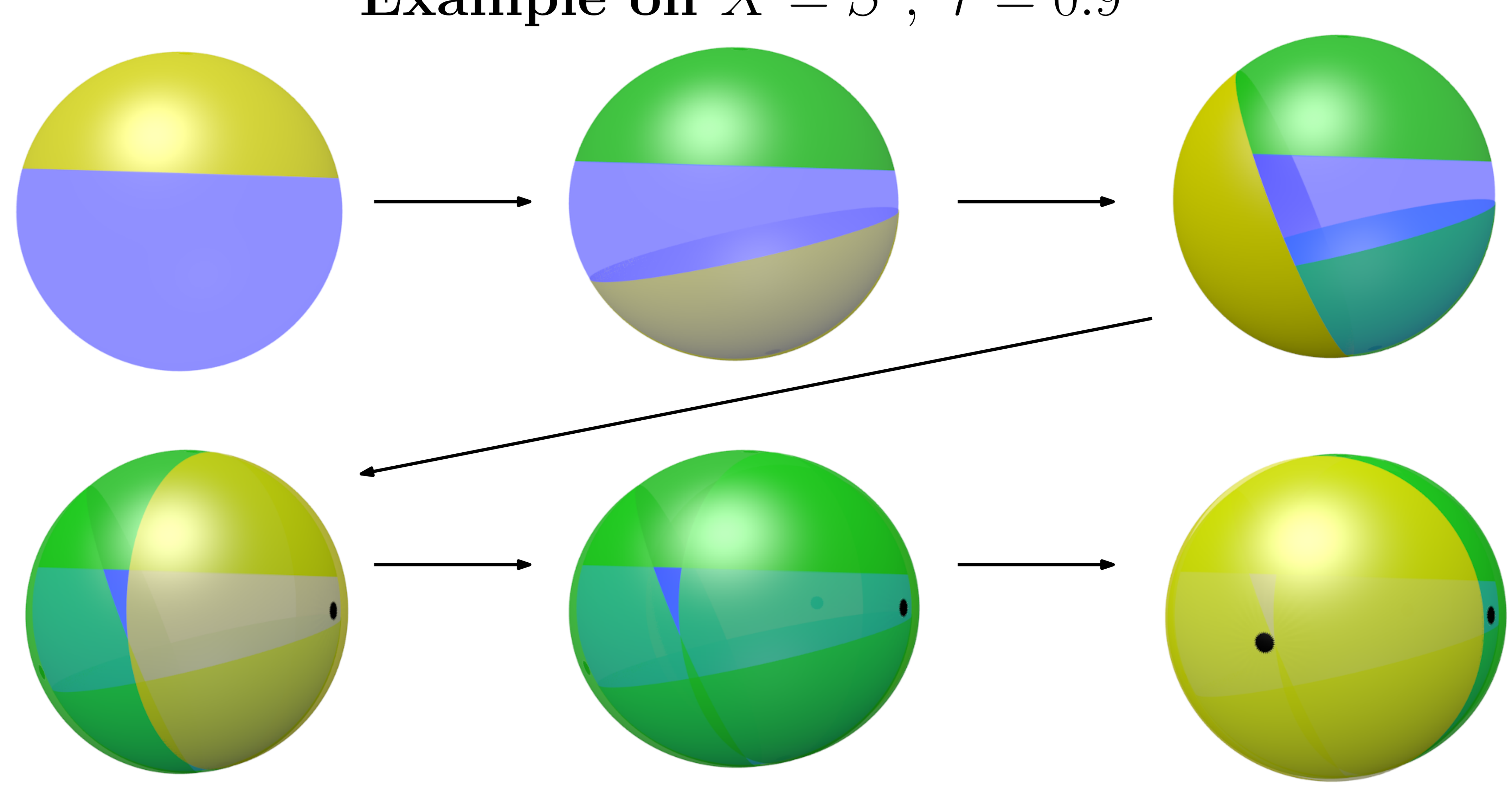
Theorem 1 (R., Vallentin, 2017)

For every $\varepsilon > 0$ with $\frac{r}{2} > \varepsilon > 0$ the covering number satisfies

$$\frac{1}{\omega_r} \leq \mathcal{N}(X, r) \leq \frac{1}{\omega_{r-\varepsilon}} \left(\ln\left(\frac{\omega_{r-\varepsilon}}{\omega_\varepsilon}\right) + 1\right).$$

Scalar ε has an optimum > 0 depending on ω and r.

Example on $X = S^2, \ r = 0.9$



Corollaries

Theorem 1 gives (after adjusting ε) as corollaries uniform proofs for:

Corollary 1 (Böröczky and Wintsche, 2003)

For $n \geq 3$ the covering density of the n-dimensional sphere by spherical balls is at most

$$n \ln n + n \ln \ln n + n + o(n).$$

Corollary 2 (Fejes Tóth, 2009)

For $n \geq 3$ the covering density of the n-dimensional Euclidean space by congruent balls is at most

$$n \ln n + n \ln \ln n + n + o(n).$$

Corollary 3 (Naszódi, 2014)

Let $K \subseteq \mathbb{R}^n$ be a bounded measurable set. Then there is a covering of $\mathbb{R}^n$ by translated copies of K of density at most

$$\inf_{\delta > 0} \left\{ \frac{\omega(K)}{\omega(K_{-\delta})} \left(\ln\left(\frac{\omega(K_{-\delta/2})}{\omega(B(0, \delta/2))} \right) + 1 \right) \right\},$$

where $K_{-\delta} = \{x \in K : \ B(x, \delta) \subseteq K\}$ is the δ-inner parallel body of K, assumed to be nonempty.

The bounds of Corollaries 1 & 3 could even slightly be improved.

Future Work

Lower bounds improving $\frac{1}{\omega_r}$

- using Lasserre hierarchy

Further Applications in:

- Probability Theory ("metric entropy")
- Definition given by Kolmogorov
- Estimate bounds of Gaussian processes