April 17th, 2013

Algebraic Number Theory II
Exercise Sheet 2

Exercise 2.1 (2 Points)**.** Let A be an integral domain. An element $a \neq 0$ is called *irreducible* if it is not a unit, and if whenever one can write $a = bc$ with $b, c \in A$ then b or c is a unit.

Now consider the ring $A = \mathbb{Z}[\sqrt{-6}]$, i.e. $A = \mathbb{Z} \oplus \sqrt{-6}\,\mathbb{Z}$ as a $\mathbb{Z}$-module, endowed with the ring structure induced by the ring structure of $\mathbb{C}$ via the obvious injective homomorphism $\mathbb{Z} \oplus \sqrt{-6}\,\mathbb{Z} \hookrightarrow \mathbb{C}$. ($A$ is the integral closure of $\mathbb{Z}$ in the number field $K = \mathbb{Q}(\sqrt{-6})$, but this is not part of the exercise.) Show that in A the number -6 has two different factorizations as products of irreducible elements:

$$-6 = -2 \cdot 3 = (\sqrt{-6})^2.$$

Exercise 2.2 (4 Points)**.** Let A be a Dedekind domain.

(a) Let $\mathfrak{a} \neq 0$ be an ideal in A. Show that every ideal in $A/\mathfrak{a}$ is principal.
(*Hint:* For $\mathfrak{a} = \mathfrak{p}^n$ the power of a prime ideal the only proper ideals of $A/\mathfrak{a}$ are $\mathfrak{p}/\mathfrak{p}^n, \dots, \mathfrak{p}^{n-1}/\mathfrak{p}^n$. Choose $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$ and show $\mathfrak{p}^\nu = A\pi^\nu + \mathfrak{p}^n$. Reduce to this case by means of the Chinese Remainder Theorem (which you do not need to prove): Let A be a ring and $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ relatively prime ideals of A, i.e. $\mathfrak{a}_i + \mathfrak{a}_j = A$ for all $i \neq j$. Then the canonical map of A onto $A/\mathfrak{a}_i$ for each factor induces an isomorphism $A/\bigcap \mathfrak{a}_i \xrightarrow{\sim} \prod A/\mathfrak{a}_i$.)

(b) Deduce that every ideal in A can be generated by at most two elements.

Exercise 2.3 (4 Points)**.** Let A be an integral domain. An element $a \neq 0$ is said to have a *unique factorization into irreducible elements* if there exist a unit u and irreducible elements p_i ($i = 1, \dots, r$) in A such that $a = u \prod_{i=1}^r p_i$, and if given two factorizations into irreducible elements $a = u \prod_{i=1}^r p_i = v \prod_{i=1}^s q_i$, we have $r = s$, and after permutation of the indices i, we have $p_i = u_i q_i$ for some unit $u_i \in A$, $i = 1, \dots, r$. An integral domain A is called *factorial* if every element $\neq 0$ has a unique factorization into irreducible elements (here we adopt the convention that a unit of A has a factorization into irreducible elements, taking $r = 0$).

Now let A be a Dedekind domain. Show that A is factorial if and only if A is a principal ideal domain.

Exercise 2.4 (6 Points)**.** Let A be a Dedekind domain with field of fractions $K = \mathrm{Frac}(A)$, let $L \mid K$ be a finite field extension and B the integral closure of A in L.

(a) Let $\mathfrak{p}$ be a prime ideal of A. Show that $\mathfrak{p}B \neq B$.
(*Hint:* You may assume $\mathfrak{p} \neq 0$. Choose $\pi \in \mathfrak{p} \backslash \mathfrak{p}^2$, write $\pi A = \mathfrak{p}\mathfrak{a}$ with $\mathfrak{p} \nmid \mathfrak{a}$. Write $1 = p + a$ for some $p \in \mathfrak{p}, a \in \mathfrak{a}$ and deduce $a \notin \mathfrak{p}$. On the other hand, using $B \cap K = A$ show that $B = \mathfrak{p}B$ implies $a \in \mathfrak{p}$.)

(b) Let $\mathfrak{a}$ be an ideal of A. Show that $\mathfrak{a} = \mathfrak{a}B \cap A$.

(c) Let $\mathfrak{a}$ and $\mathfrak{a}'$ be ideals of A. Show $\mathfrak{a} \mid \mathfrak{a}' \iff \mathfrak{a}B \mid \mathfrak{a}'B$.