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Number Theory I (Commutative Algebra)

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Exercise sheet 11

As in the textbook, all rings are commutative with a unit.

If you submit solutions, please choose only a few exercises.

Exercise 1. Let K be a field equipped with a surjective function $v : K^* \to \mathbb{Z}$ which is a discrete valuation, $A = v^{-1}\mathbb{Z}_{\geq 0}$ its associated dvr, and $\mathfrak{m} = v^{-1}\mathbb{Z}_{\geq 1}$ the maximal ideal of A. Show that A has depth one (no theorems are needed; just the definitions).

Exercise 2. Let R be a Noetherian local ring, and M a finitely generated R-module. Recall that Show that $\mathfrak{m} \in \mathrm{Ass}(M)$ if and only if $\mathfrak{m} = \mathrm{z.div}(M)$. Hint: [AK2017, 3.12], [AK2017, 17.12], [AK2017, 17.17]

Recall that an element is called *nilpotent* if $a^n = 0$ for some $n \in \mathbb{N}$, and a ring A is said to be *reduced* if it has no nonzero nilpotents.

Exercise 3 ([AK2017, Ex.13.57])**.** Let A be a ring.

(1) Show that if A is reduced, then so is $S^{-1}A$ for any multiplicative set S.
(2) Show that A is reduced if $A_\mathfrak{p}$ is reduced for all primes $\mathfrak{p}$ (in fact it suffices that $A_\mathfrak{m}$ is reduced for all maximal ideals $\mathfrak{m}$).
(3) Let $\mathrm{nil}(A)$ denote the set of nilpotent elements of a ring A. Show that $\mathrm{nil}(A) = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p}$.

Exercise 4 ([AK2017, 23.12])**.** Let A be a Noetherian ring. We will show that A is reduced if and only if it satisfies (R_0) and (S_1).

(1) Suppose A is reduced. Show that (R_0) is satisfied (cf. Exercise 2(1) above).
(2) Suppose A is reduced, and $\mathfrak{p}$ is a prime of height ≥ 1. Show that we have $\mathfrak{p}A_\mathfrak{p} \notin \mathrm{Ass}(A_\mathfrak{p})$ (cf. Exercise 2(1) above). Deduce that depth $A_\mathfrak{p} \geq 1$, and that (S_1) is satisfied.
(3) Recall that
    (a) $A \to \prod_{\mathfrak{p} \in \mathrm{Ass}(A)} A_\mathfrak{p}$ is injective.
    (b) (S_1) is satisfied $\Leftrightarrow$ $\mathrm{Ass}(A)$ is the set of minimal primes of A.
    Assuming A satisfies (R_0) and (S_1) show that A is reduced.

We did not discuss the following proposition, but we will use it in the next exercise.

Proposition ([AK2017, 23.13])**.** *Let A be a Noetherian domain. Then*

$$A = \bigcap_{\{\mathfrak{p} \in \mathrm{Spec}(A) \ | \ \mathrm{depth}(A_\mathfrak{p}) = 1\}} A_\mathfrak{p}$$

where the intersection happens in $\mathrm{Frac}(A)$*.*

Exercise 5 ([AK2017, 23.15])**.** Let A be a Noetherian domain. Recall that in class, we showed that A normal implies A satisfies (R_1) and (S_2). Here we will show the converse.

(1) Suppose that A satisfies (R_1) and let $a/b \in \mathrm{Frac}(A)$ be an integral element. Show that $a/b \in A_\mathfrak{p}$ for every $\mathfrak{p}$ of height one.

(2) Suppose that A satisfies (R_1) and (S_2). Use [AK2017, 23.13] (above) to show that A is normal.