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Number Theory I (Commutative Algebra)

Hélène Esnault, Exercises: Shane Kelly

Exercise sheet 7

As in the textbook, all rings are commutative with a unit.

If M' is an R'-module for some ring R', recall that

$$\mathrm{Spec}(R') \overset{def}{=} \{\text{prime ideals of } R'\} \qquad (\text{cf. [AK2017, 13.1]}),$$

$$\mathrm{Ann}(M') \overset{def}{=} \{x \in R' \mid xm = 0 \;\forall\; m \in M'\} \qquad (\text{cf. [AK2017, 4.1]}),$$

$$\mathrm{Supp}(M') \overset{def}{=} \{\mathfrak{r} \in \mathrm{Spec}(R') \mid M'_{\mathfrak{r}} \neq 0\} \qquad (\text{cf. [AK2017, 13.3]}),$$

and that we have

$$\mathrm{Supp}(M') \subset \{\mathfrak{r} \in \mathrm{Spec}(R') \mid \mathrm{Ann}(M') \subset \mathfrak{r}\}$$

with equality if M' is finitely generated, [AK2017, 13.4(3)].

Exercise 1 ([AK2017, Theorem 14.8], Going-down for flat modules)**.** Let $\phi : R \to R'$ be a map of rings, M' a finitely generated R'-module, $\mathfrak{p} \subset \mathfrak{q}$ nested primes of R, and $\mathfrak{q}'$ a prime of $\mathrm{Supp}(M')$ lying over $\mathfrak{q}$. Assume M' is flat over R. We will show that there is a prime $\mathfrak{p}' \in \mathrm{Supp}(M')$ lying over $\mathfrak{p}$ and contained in $\mathfrak{q}'$.

$$\begin{array}{ccccccc} \mathfrak{p}' & \subset & \mathfrak{q}' & \in & \mathrm{Supp}(M') & \subset & \mathrm{Spec}(R') \\ & & & & & & \big\downarrow {\scriptstyle \mathfrak{r} \mapsto \phi^{-1}\mathfrak{r}} \\ \mathfrak{p} & \subset & \mathfrak{q} & & & \in & \mathrm{Spec}(R) \end{array}$$

(1) Replacing R, R', and M' with $R/\mathfrak{p}$, $R'/\mathfrak{p}R'$ and $M'/\mathfrak{p}M'$, show that we can assume $\mathfrak{p} = \langle 0 \rangle$ and R is integral. (I.e.,
   (a) Show that $M'/\mathfrak{p}M'$ is a finitely generated $R'/\mathfrak{p}R'$-module,
   (b) $M'/\mathfrak{p}M'$ is a flat $R/\mathfrak{p}$-module, and that
   (c) if we can find a prime $\mathfrak{p}'' \in \mathrm{Supp}(M'/\mathfrak{p}M')$ lying over (0) and containing $\mathfrak{q}'/\mathfrak{p}\mathfrak{q}'$, then we can find a prime $\mathfrak{p}' \in \mathrm{Supp}(M')$ lying over $\mathfrak{p}$ and contained in $\mathfrak{q}'$.)

(2) Replacing R' with $R'/\mathrm{Ann}(M')$ and $\mathfrak{q}'$ with $\mathfrak{q}'/\mathrm{Ann}(M')$, show that we can assume that we have $\mathrm{Ann}(M') = 0$. (I.e., show that M' is a finitely generated $R'/\mathrm{Ann}(M')$-module and if we can find a prime $\mathfrak{p}'' \in \mathrm{Supp}_{R'/\mathrm{Ann}(M')}(M')$ lying over (0) and contained in $\mathfrak{q}'/\mathrm{Ann}(M')$, then we can find a prime $\mathfrak{p}' \in \mathrm{Supp}_{R'}(M')$ lying over $\mathfrak{p}$ and contained in $\mathfrak{q}'$).

(3) Show that we have $\mathrm{Supp}(M') = \mathrm{Spec}(R')$, since M' is finitely generated and we are now assuming $\mathrm{Ann}(M') = 0$. Hence, any prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$ and contained in $\mathfrak{q}'$ satisfies our requirements.

(4) Use Zorn's lemma to find any minimal prime, say $\mathfrak{p}'$ of R' contained in $\mathfrak{q}'$.

(5) Use the localisation $R'_{\mathfrak{p}'}$ at our minimal prime $\mathfrak{p}'$ and

$$\sqrt{\langle 0 \rangle} = \bigcap_{\text{primes } \mathfrak{r} \in \mathrm{Spec}(R'_{\mathfrak{p}'})} \mathfrak{r} \qquad \text{[AK2017, Theorem 3.14]}$$

to show that all elements of the chosen minimal prime $\mathfrak{p}'$ are zerodivisors of R'.

(6) Choosing generators for M', and recalling that we are now assuming we have $\mathrm{Ann}(M') = 0$, find an injective R'-module morphism $R' \to M'^n$ for some n.

(7) By considering the maps "multiplication by x" maps

$$\mu_x : R \to R,$$
$$\mu_x : M' \to M',$$
$$\mu_x : M'^n \to M'^n,$$
$$\mu_x : R' \to R'$$

for an element x of our now-assumed-integral-ring R, and parts (5) and (6), show that $\mathfrak{p}' \cap R = \langle 0 \rangle$.

Note, since we are now assuming $\mathfrak{p} = \langle 0 \rangle$, and $\mathrm{Supp}(M') = \mathrm{Spec}(R')$, and we chose $\mathfrak{p}'$ to be contained in $\mathfrak{q}'$, we are finished.