January 6, 2017

Étale Cohomology

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Exercise sheet 10[1]

Exercise 1. Let F be a presheaf on $X_{\text{ét}}$. Show that F is a sheaf if and only if it satisfies

(1) For each $U \to X$ in $X_{\text{ét}}$, the association that to each Zariski open V of U we associate the value $F(V \subseteq U \to X)$ defines a Zariski sheaf U_{Zar}.
(2) For each étale morphism $U \to X$ with U affine and each cover $\{U_i \to U\}_{i \in I}$ with U_i affine the following sequence:

$$F(U) \to \Pi_{i \in I} F(U_i) \rightrightarrows \Pi_{i,j \in I} F(U_i \times_U U_j)$$

is exact.

Exercise 2. Let X be a scheme, and let F be a quasi-coherent sheaf on X. We define $W(F)$ to be the presheaf on $X_{\text{ét}}$ which associates any étale map $u : U \to X$ the $\Gamma(U, \mathcal{O}_U)$-module $\Gamma(U, u^*F)$. Show that $W(F)$ is a sheaf on $X_{\text{ét}}$.

Exercise 3. Let X be a scheme, and let S be any set. Show that the association $F : (U \to X) \mapsto \text{Hom}_{\text{cont}}(|U|, S)$, where $(U \to X) \in X_{\text{ét}}$, $|U|$ denotes the underlying topological space of U, S is considered as a topological space with discrete topology, and Hom_{cont} denote the set of continuous maps, is an étale sheaf. This sheaf F is called the constant sheaf with constant value S. It is the sheaf associated to the constant presheaf $(U \to X) \mapsto S$.

Exercise 4. Let X be a scheme over S, and $\Omega^1_{X/S}$ is the sheaf of differentials. We define $W(\Omega^1_{X/S})$ to be the presheaf on $X_{\text{ét}}$ which associates any étale map $u : U \to X$ the $\Gamma(U, \mathcal{O}_U)$-module $\Gamma(U, u^*\Omega^1_{X/S})$. Show that $\Gamma(U, W(\Omega^1_{X/S})) = \Gamma(U, \Omega^1_{U/S})$. In particular $W(\Omega^1_{X/S})$ is a sheaf on $X_{\text{ét}}$.

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[1]If you want your solutions to be corrected, please hand them in just before the lecture on January 11, 2017. If you have any questions concerning these exercises you can contact Shane Kelly via shanekelly64@gmail.com or Lei Zhang via l.zhang@fu-berlin.de.

Exercise 5. Let (E, J) be a site. Show that the topology J is always defined by a pretopology.

Exercise 6. Let S be a scheme. Let $\mathcal{F}$ be a presheaf, i.e. a functor $(\mathrm{Sch}/S) \to (\mathrm{Sets})$. In the following we are going to define, step by step, the sheafication of $\mathcal{F}$.

(1) Let $U \in (\mathrm{Sch}/S)$. Take $\mathcal{F}^s(U)$ to be the set $\mathcal{F}(U)/\sim$, where $\sim$ is an equivalence relation defined as follows: If $a, b \in \mathcal{F}(U)$, then $a \sim b$ if and only if there is a covering $\{U_i \to U\}_{i \in I}$ such that $a|_{U_i} = b|_{U_i}$ for all $i \in I$. Show that in this way we get a presheaf $\mathcal{F}^s$ which is separated, i.e. for any covering $\{U_i \to U\}_{i \in I}$ the map $\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$ is injective.
(2) Let $U \in (\mathrm{Sch}/S)$. Take $\mathcal{F}^a(U)$ to be the set of pairs $(\{U_i \to U\}_{i \in I}, \{a_i\}_{i \in I})$, where $\{U_i \to U\}_{i \in I}$ is a covering and $a_i \in \mathcal{F}^s(U_i)$ such that the pullback of a_i, a_j to $\mathcal{F}^s(U_i \times_U U_j)$ concide, modulo the following equivalent relation: $(\{U_i \to U\}_{i \in I}, \{a_i\}_{i \in I})$ is equivalent to $(\{V_j \to U\}_{j \in J}, \{b_j\}_{j \in J})$ if and only if the restriction of a_i, b_j to $\mathcal{F}^s(U_i \times_U V_j)$ coincide. Show that F^a is a sheaf.
(3) Show that the composition $\pi : \mathcal{F} \to \mathcal{F}^s \to \mathcal{F}^a$ satisfies the following universal property: Given any fppf sheaf $\mathcal{G}$ and any map $\phi : \mathcal{F} \to \mathcal{G}$ there exists a unique map $\lambda : \mathcal{F}^a \to \mathcal{G}$ such that $\phi = \lambda \circ \pi$.
(4) Show that $\mathcal{F} \to \mathcal{F}^a$ is injective if and only if $\mathcal{F}$ is separated.