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Étale Cohomology

Dr. Lei Zhang

Exercise sheet 1[1]

Exercise 1. Let $f : X \to \operatorname{Spec}(A)$ be a morphism of schemes. Let $a \in A$ be an element. We know that $U := \operatorname{Spec}(A_a) \subseteq \operatorname{Spec}(A)$ is an open subset. Show that $\Gamma(f^{-1}(U), \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)_a$.

Exercise 2. Recall that a morphism of schemes $i : Y \hookrightarrow X$ is called a closed embedding if the topological space of Y is mapped homoeomorphically by i to a closed sub topological space of X, and if the map of sheaf of rings $\mathcal{O}_X \to i_*\mathcal{O}_Y$ is surjective. Show that a closed embedding is finite.

Exercise 3. Show that the composition of two affine (resp. finite) morphisms is still affine (resp. finite). Show also that the base change of an affine (resp. a finite) morphism is affine (resp. finite).

Exercise 4. Let $f : X \to Y$ be a morphism of schemes. We say f is *of finite type* if there exists an affine covering $\{U_i\}_{i\in I}$, where $U_i = \operatorname{Spec}(A_i)$, of Y such that for each U_i, $f^{-1}(U_i)$ is covered by finitely many open affines $\{V_j\}_{1\leq j\leq n_i}$, where $V_j = \operatorname{Spec}(B_j)$, and the induced map $A_i \to B_j$ makes B_j a finitely generated A_i-algebra, i.e. $B_j = A_i[x_1, \cdots, x_s]$ with $x_1, \cdots, x_s \in B_j$. Show that if f is a finite type morphism of schemes then for any affine open $U = \operatorname{Spec}(A) \subseteq Y$, $f^{-1}(U)$ is covered by finitely many open affines $\{V_j = \operatorname{Spec}(B_j)\}_{1\leq j\leq n_i}$ such that the induced map $A_i \to B_j$ makes B_j a finitely generated A_i-algebra.

Exercise 5. Let k be a field. Let $X \to \operatorname{Spec}(k)$ be a morphism of finite type. Then the following statements are equivalent.

(1) X is affine and $\Gamma(X, \mathcal{O}_X)$ is an artinian ring;
(2) $X \to \operatorname{Spec}(k)$ is finite;
(3) The underlying topological space of X is discrete.

(Hint: You can use Atiyah, Macdonald, *Introduction to commutative algebra*, Chapter 8, Exercise 3, pp. 92.)

[1] If you want your solutions to be corrected, please hand them in just before the lecture on October 26, 2016. If you have any questions concerning these exercises you can contact Shane Kelly via shanekelly64@gmail.com or Lei Zhang via l.zhang@fu-berlin.de.