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Number Theory I

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Exercise sheet 4[1]

Exercise 1. Prove the following statements:

(i) For any $n \in \mathbb{Z}$, there is an isomorphism of $\mathbb{Z}$-modules
$$(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}.$$

(ii) If $n, m \in \mathbb{Z}$ are coprime, i.e. if they do not have a common prime divisor, then $(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) = 0$.

Exercise 2. We consider $\mathbb{Q}$ as $\mathbb{Z}$-module, and for every prime number p the $\mathbb{Z}$-module $\mathbb{Z}/p\mathbb{Z}$. Let P be the set $\{p \in \mathbb{Z} \mid p \,\text{is a prime number}\}$, we consider the following two $\mathbb{Z}$-modules

(i) $\mathbb{Q} \otimes_{\mathbb{Z}} (\prod_{p \in P} \mathbb{Z}/p\mathbb{Z})$

(ii) $\prod_{p \in P}(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z})$

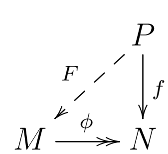
Are these two $\mathbb{Z}$-modules isomorphic?

Exercise 3. Let A be a ring, I an ideal contained in the Jacobson radical of A; let M be an A-module and N a finitely generated A-module, and let $u : M \to N$ be a homomorphism. If the induced homomorphism $M/IM \to N/IN$ is surjective, then u is surjective.

Exercise 4. Let A be a commutative ring and P an A-module. Show that the following properties are equivalent:

(i) For any short exact sequence
$$0 \to L \to M \to N \to 0$$
the associated sequence
$$0 \to \mathrm{Hom}_A(P, L) \to \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, N) \to 0$$
is also exact.

(ii) For any surjective morphism $\phi : M \to N$ of A-modules, and any A-linear morphism $f : P \to N$, there exists a morphism $F : P \to N$ lifting f, i.e. such that the diagram



[1] If you want your solutions of this exercises to be corrected, please hand them in before the exercise class on November 13th.

commutes.

(iii) Every short exact sequence of A-modules

$$0 \to L \to M \xrightarrow{\tau} P \to 0$$

splits, i.e. there exists an A-module homomorphism $\sigma : P \to M$, such that $\tau \circ \sigma = \mathrm{id}_P$.

(iv) P is a direct summand of a free A-module.

If P satisfies the above equivalent properties, then P is called *projective.*

Exercise 5. Use Nakayama's Lemma to show that if A is a local ring, i.e. if A has precisely one maximal ideal, and if P is a finitely generated projective A-module, then P is free of finite rank.

References

[AM69] M. F Atiyah and I. G Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., 1969.