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Remark on Exercise sheet 3

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Remark 1. Exercise 1 of exercise sheet 3 was not stated properly. The correct way to state it is the following: Let A be a ring and let I be an ideal of A, and let Λ be a non empty set, and let M_λ be an A-module for each $\lambda \in \Lambda$. Prove that if $M_\lambda \subseteq M$ with M an A-module then $I(\sum_\lambda M_\lambda) = \sum_\lambda (IM_\lambda)$.

Proof. We prove first that $I(\sum_\lambda M_\lambda) \subseteq \sum_\lambda (IM_\lambda)$.

We take an element $\alpha \in I(\sum_\lambda M_\lambda)$, then $\alpha = \sum_{i=1}^n a_i m_i$ with $a_i \in I$ and $m_i \in \sum_\lambda M_\lambda$. Then there exist elements $n_{i,\lambda} \in M_\lambda$ such that $m_i = \sum_\lambda n_{i,\lambda}$ with only a finite number of $n_{i,\lambda}$ different from 0. Hence

$$\alpha = \sum_{i=1}^n a_i m_i = \sum_{i=1}^n a_i (\sum_\lambda n_{i,\lambda}) = \sum_\lambda (\sum_{i=1}^n a_i n_{i,\lambda})$$

and from the last expression we can see that α is an element of $\sum_\lambda (IM_\lambda)$.

We prove now that $\sum_\lambda (IM_\lambda) \subseteq I(\sum_\lambda M_\lambda)$.

We take an element $\beta \in \sum_\lambda (IM_\lambda)$, then $\beta = \sum_\lambda m_\lambda$ such that $m_\lambda \in IM_\lambda$ and $m_\lambda \neq 0$ only for a finite number of indices. Since $m_\lambda \in IM_\lambda$ then $m_\lambda = \sum_{i=1}^{k_\lambda} a_{i,\lambda} m_{i,\lambda}$ with $a_{i,\lambda} \in I$ and $m_{i,\lambda} \in M_\lambda$. Hence

$$\beta = \sum_\lambda m_\lambda = \sum_\lambda \sum_{i=1}^{k_\lambda} a_{i,\lambda} m_{i,\lambda}$$

But for every pair (i, λ) the element $m_{i,\lambda}$ is in $M_\lambda \subseteq \sum_\lambda M_\lambda$, hence $a_{i,\lambda} m_{i,\lambda} \in IM_\lambda \subseteq I(\sum_\lambda M_\lambda)$, hence since

$$\beta = \sum_\lambda (\sum_{i=1}^{k_\lambda} a_{i,\lambda} m_{i,\lambda})$$

it is a finite sum of elements of $I(\sum_\lambda M_\lambda)$, hence β is an element of $I(\sum_\lambda M_\lambda)$. $\square$

Remark 2. Here is an example of a finitely generated A-module M, which is not a ring, and of a sub A-module $M' \subseteq M$ which is not finitely generated. Let us take A be the polynomial ring with infinitely variables with complex coefficients $\mathbb{C}[X_1, X_2, \dots]$, we consider the free A-module $M = (X_1)$, then the submodule $M' = (X_1X_2, X_1X_3, X_1X_4, \dots, X_1X_n, \dots)$ is a sub A-module of M which is not finitely generated.