October 20th, 2015

Number Theory I

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Exercise sheet 2[1]

Exercise 1. Let A be a ring and $A[X]$ the polynomial ring in one variable X with coefficients in A.

(i) Prove that in $A[X]$ the Jacobson radical is equal to the nilpotent radical. (*Hint:* Use proposition 1.9 of [AM69].)
(ii) Show with an example that the Jacobson radical of A is not in general equal to the nilpotent radical of A.

Exercise 2. (i) Let A be a ring and I an ideal of A. Let $\pi : A \to A/I$ the natural projection. Prove that the correspondence induced by π between ideals of A which contain I and ideals of A/I preserves maximals ideals.
(ii) Let $n \geq 1$ and let p be a prime number. Deduce from this that $\mathbb{Z}/p^n\mathbb{Z}$ is a local ring (*i.e.* a ring with exactly one maximal ideal).
(iii) For which $n \geq 1$ the ring $\mathbb{Z}/p^n\mathbb{Z}$ is a integral domain?

Exercise 3. Let A be an integral domain and $A[X, Y]$ the polynomial ring in two variables with coefficients in A. Let $m, n \in \mathbb{Z}_{\geq 1}$ be positive integers.

Show that the ideal $(X^m - Y^n)$ is prime in $A[X, Y]$ if and only if m and n are coprime, i.e. $(m, n) = (1)$.
(*Hint:* For the "if" direction: Show that the map $\varphi : A[X, Y] \to A[T]$, $f(X, Y) \mapsto f(T^n, T^m)$ is a ring homomorphism, which factors over a ring homomorphism $\bar{\varphi} : A[X, Y]/(X^m - Y^n) \to A[T]$. Then show that $\bar{\varphi}$ is injective and conclude with Ex. 2 of the first sheet.)

Exercise 4. For a ring A we denote by $\operatorname{Spec} A$ the set of prime ideals in A, it is called the *spectrum of* A. For every ideal I we define by $V(I) = \{P \in \operatorname{Spec} A \mid I \subset P\}$. Prove the following properties:

(i) $V(0) = \operatorname{Spec} A$ and $V(1) = \emptyset$,
(ii) If I is an ideal of A, then $V(I) = V(\sqrt{I})$
(iii) If I and J are ideals of A, such that $I \subset J$, then $V(J) \subset V(I)$
(iv) If I and J are ideals of A, $V(I \cap J) = V(IJ) = V(I) \cup V(J)$,
(v) If $\{I_\alpha\}_\alpha$ is a family of ideals in A, then $V(\sum_\alpha I_\alpha) = \cap_\alpha V(I_\alpha)$

[1] If you want your solutions of this exercises to be corrected, please hand them in before the exercise class on October 30th.

(vi) Remark that the sets $V(I)$ verifies the axioms for closed sets in a topological space, the resulting topology is called Zariski topology on $\operatorname{Spec} A$.

Exercise 5. Let $f : A \to B$ be a ring homomorphism.

(i) Prove that this induces a map $f^{-1} : \operatorname{Spec} B \to \operatorname{Spec} A$, $\mathfrak{p} \mapsto f^{-1}(\mathfrak{p})$.

(ii) Let $A_1, \ldots, A_n$ be rings, denote by $A = A_1 \times \ldots \times A_n$ their product and by $\pi_i : A \to A_i$, $(a_1, \ldots, a_n) \mapsto a_i$, $i = 1 \ldots, n$, the projection maps.

Show that $\pi_i^{-1} : \operatorname{Spec} A_i \to \operatorname{Spec} A$ maps bijectively onto $\pi_i^{-1}(\operatorname{Spec} A_i)$ and that we have the following decomposition of $\operatorname{Spec} A$ into disjoint sets

$$\operatorname{Spec} A = \pi_1^{-1}(\operatorname{Spec} A_1) \sqcup \ldots \sqcup \pi_n^{-1}(\operatorname{Spec} A_n) \xleftarrow{\text{bij.}} \operatorname{Spec} A_1 \sqcup \ldots \sqcup \operatorname{Spec} A_n.$$

(iii) Let $\pi : A \to A_{\text{red}} = A/\text{nil(A)}$ be the canonical surjection. Show that $\pi^{-1} : \operatorname{Spec} A_{\text{red}} \to \operatorname{Spec} A$ is bijective.

References

[AM69] M. F Atiyah and I. G Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., 1969.