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Number Theory I

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Exercise sheet 11[1]

Exercise 1. Let A be a subring of B, such that $B \setminus A$ is closed under multiplication. Show that A is integrally closed in B.

Exercise 2. Let $A \subseteq B$ be an imbedding of integral domains, $S := A \setminus \{0\}$. If B is integral over A, then $S^{-1}B$ is a field.

Exercise 3.

(i) Show that $R = \mathbb{Z}[\frac{-1+\sqrt{-3}}{2}]$ is an Euclidean domain. (Hint: First show that for any $\theta \in \mathbb{Q}[\sqrt{-3}]$ you can find $\theta' \in R$ such that $|\theta - \theta'| < 1$. Then show that if $\alpha, \beta \in R$ with $0 < |\beta| < |\alpha|$, one can find $\gamma \in R$ such that $|\alpha - \beta\gamma| < |\beta|$.)

(ii) Is $\mathbb{Z}[\sqrt{-3}]$ integrally closed? Justify your answer.

Exercise 4. Recall that a ring homomorphism $f : R \to A$ is called faithfully flat if it is flat and for any R-module M, $M \otimes_R A = 0$ implies that $M = 0$. Let $f : R \to A$ be any ring homomorphism, show that the following conditions on f are equivalent.

(i) The map f is faithfully flat;

(ii) A sequence $N' \to N \to N''$ of R-modules is exact if and only if $N \otimes_R A \to N' \otimes_R A \to N'' \otimes_R A$ is exact;

(iii) The map f is flat, and for any maximal ideal $p \subseteq R$, $pA \neq A$;

(iv) The map f is flat, and for any prime ideal $p \subseteq R$, there exists a prime ideal $P \subseteq A$ such that $f^{-1}(P) = p$.

In the end conclude that if f is flat injective, and if A is integral over R then f is faithfully flat.

[1]If you want your solutions of this exercise to be corrected, please hand them in before the exercise class on January 15th.