November 26, 2013

Algebra I

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Exercise sheet 7[1]

Exercise 1. Let A be a commutative ring and write $\operatorname{Spec} A$ for the set of prime ideals of A. For any ideal $I \subset A$ write $V(I)$ for the set of prime ideals $\mathfrak{p} \in \operatorname{Spec} A$ containing I. If $\phi : A \to B$ is a homomorphism of commutative rings, we know that $\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$ induces a map $\phi^* : \operatorname{Spec} B \to \operatorname{Spec} A$.

(a) If $S \subset A$ is a multiplicatively closed subset, and $\phi : A \to S^{-1}A$ the localization morphism, show that ϕ^* is injective.

(b) If $f \in A$, write $S_f := \{f^n, n \geq 0\}$, where $f^0 := 1$. This is a multiplicatively closed subset of A; again write $\phi : A \to S_f^{-1}A$ for the localization map. Show that $\phi^* : \operatorname{Spec} S_f^{-1}A \to \operatorname{Spec} A$ is injective and that its image is $(\operatorname{Spec} A) \setminus V((f))$. What happens if f is nilpotent?

Exercise 2. Let A be an integral domain and M an A-module. An element $m \in M$ is called *torsion element, or A-torsion element* if there exists $a \in A \setminus \{0\}$ such that $am = 0$. Write $T_A(M)$ for the set of A-torsion elements. Prove that:

(a) $T_A(M)$ is an A-submodule of M.

(b) For any multiplicatively closed set $S \subset A$, there is an isomorphism of $S^{-1}A$-modules $S^{-1}T_A(M) \xrightarrow{\cong} T_{S^{-1}A}(S^{-1}M)$.

(c) The following statements are equivalent:

 (i) $T_A(M) = 0$.

 (ii) $T_{A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ for all prime ideals $\mathfrak{p} \subset A$.

 (iii) $T_{A_\mathfrak{m}}(M_\mathfrak{m}) = 0$ for all maximal ideals $\mathfrak{m} \subset A$.

(d) Now assume that A contains zero-divisors. Is $T_A(M) \subset M$ still an A-submodule? If not, find a counterexample.

Exercise 3. Recall from problem set 4 the definition of a projective module. Let A be a commutative ring and M an A-module. Consider the following statements:

(a) M is a projective A-module.

(b) $M_\mathfrak{p}$ is a projective $A_\mathfrak{p}$-module for all prime ideals $\mathfrak{p} \subset A$.

[1] If you want your solutions of this exercise to be corrected, please hand them in just before the lecture on December 3. Questions or comments to kindler@math.fu-berlin.de or come to A3, Room 112A.

(c) $M_{\mathfrak{m}}$ is a projective $A_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m} \subset A$.
(d) $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of A.

Show that (a)⇒(b)⇒(c). If M is finitely generated, show that (c)⇒(d).

If $A = \mathbb{Z}$, show that if M is finitely generated, then (d)⇒(a).

Exercise 4. Let A be a commutative ring and M an A-module. Define the *support* $\operatorname{supp}_A(M)$ *of* M to be the set of prime ideals $\mathfrak{p} \subset A$ such that $M_{\mathfrak{p}} \neq 0$. Prove the following statements:

(a) $M \neq 0$ if and only if $\operatorname{supp}_A(M) \neq \emptyset$.
(b) If $I \subset A$ is an ideal, then $\operatorname{supp}_A(A/I) = V(I)$.
(c) If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of A-modules, then $\operatorname{supp}_A(M) = \operatorname{supp}_A(M') \cup \operatorname{supp}_A(M'')$.
(d) If N, M are finitely generated A-modules, then $\operatorname{supp}_A(M \otimes_A N) = \operatorname{supp}_A(M) \cap \operatorname{supp}_A(N)$.