November 11th, 2013

Algebra I

Prof. H. Esnault

Exercise sheet 5[1]

Exercise 1. Let A be a commutative ring.

(i) If M, N are two flat A-modules, then so is $M \otimes_A N$.
(ii) If B is a flat A-algebra and N is a flat B-module, then N is flat as an A-module.
(iii) If M is a flat A-module, B is an A-algebra, then $M \otimes_A B$ is a flat B-module.
(iv) If A is a field then any A-module is flat.
(v) If $a \in A$ is a non-zero divisor (i.e. $a \neq 0$ and $a \cdot r = 0 \Rightarrow r = 0$ for any $r \in A$), and M is a flat A-module, then a is a non-zero divisor of M (i.e. $a \neq 0$ and $a \cdot m = 0 \Rightarrow m = 0$ for any $m \in M$).
(vi) If $I \subset A$ is an ideal, M is a flat A-module, then the natural map $I \otimes_A M \to IM$ sending $i \otimes m \mapsto im$ is an isomorphism.

Exercise 2. [2] Let $A := k[x, y]$ be a polynomial ring in two variables over a field k. Which of the following A-algebras are flat? Justify your answers.

(i) $k[x, y, z]$
(ii) $k[x, y]/(x, y)$
(iii) $k[x, y]/(y^2)$
(iv) $k[x, y, z]/(z^2)$
(v)* $k[x, y, z]/(xz, yz, z^2)$
(*Hint: try to write it as* $A \oplus A/(x, y)$*, then use Ex 4.2 (ii)*)

Exercise 3. Let A be a commutative ring. I, J be two ideals of A.

(i) If M is a flat A-module, then $(I \cap J)M = IM \cap JM$.
(ii) Let

$$(I : J) := \{a \in A \mid a \cdot j \in I, \forall j \in J\}.$$

[1] If you want your solutions to be corrected, please hand them in just before the lecture on Nov.19th. If you have any questions concerning these exercises you can contact Lei Zhang via l.zhang@fu-berlin.de or come to Arnimalle 3 112A.

[2] Those * exercises are not compulsory. You could do them if you want to go deeper into the subject.

Show that $(I : J) \subseteq A$ is an ideal, and if J is finitely generated, B is a flat A-algebra, then

$$(I : J)B = (IB : JB).$$

(iii) Let $A = k[x, y]$ where k is a field. Let $S = \{x^n \mid n \in \mathbb{N}\}$. Show that $S \subseteq A$ is a multiplicative closed subset.

(iv)* Let $k[x, \frac{y}{x}] \subseteq S^{-1}A$, where S is the multiplicatively closed subset of A definied in (iii). Show that $k[x, \frac{y}{x}]$ is not a flat A-algebra. (*Hint: use Ex 5.3 (i) for* $I = (x)$ *and* $J = (y)$)

Exercise 4. Let A be a commutative ring, S be a multiplicatively closed subset of A. S is called *saturated* if

$$xy \in S \quad \Leftrightarrow \quad x \in S \text{ and } y \in S.$$

(i) Let S be a saturated multiplicatively closed subset of A. Show that an element $x \in A$ is inversible in $S^{-1}A$ if and only if $x \in S$.

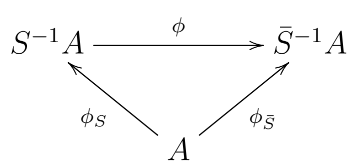
(ii) Given a prime ideal P of $S^{-1}A$ then the inverse image of P under the localization map $\phi_S : A \to S^{-1}A$ is a prime ideal p in A. Show that this defines a one to one correspondence between the prime ideals in $S^{-1}A$ and the prime ideals in A which do not meet S.

(iii) S is saturated if and only if $A \setminus S$ is a union of prime ideals.

(iv) If S is any multiplicatively closed subset of A, there is a unique smallest saturated multiplicatively closed subset $\bar{S}$ containing S, and that $\bar{S}$ is the complement in A of the union of the prime ideals which do not meet S. $\bar{S}$ is called the saturation of S.

(v) If every element $s \in S$ is inversible in A, then the localization map $\phi_S : A \to S^{-1}A$ is an isomorphism.

(vi)* Let $\bar{S}$ be the saturation of S. Show that there is a unique ring homorphism $\phi : S^{-1}A \to \bar{S}^{-1}A$ making the following diagram



commutative, and that ϕ is indeed an isomorphism.

(vii)* Let I be an ideal in A. Suppose $S := 1 + I$, i.e. the subset $\{1 + i \mid i \in I\}$ in A. Describe $\bar{S}$.