January 20th, 2014

Algebra I

Prof. H. Esnault

Exercise Sheet 13[1]

A noetherian local ring $(R, \mathfrak{m})$ is called *regular* iff $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim R$, where $k = A/\mathfrak{m}$ is the residue field.

Exercise 13.1. Let $f \in k[X_1, \ldots, X_n]$ be an irreducible polynomial over an algebraically closed field k, and $V = \{x \in k^n \mid f(x) = 0\}$ the locus of zeros of f. A point x on V is called *non-singular*, iff not all the partial derivatives $\partial f/\partial X_i$ vanish at x. Let $A = k[X_1, \ldots, X_n]/(f)$, and let $\mathfrak{m}$ be the maximal ideal of A corresponding to the point x (see Exc. 11.2 (i)). Show that x is non-singular if and only if $A_\mathfrak{m}$ is a regular local ring.

(*Hint:* By a corollary of the Dimension Theorem we have $\dim A_\mathfrak{m} = n-1$. Now (if $x = (0, \ldots, 0)$, which you can assume after a coordinate transformation)

$$\frac{\mathfrak{m}}{\mathfrak{m}^2} = \frac{(X_1, \ldots, X_n)}{(X_1, \ldots, X_n)^2 + (f)}$$

and has dimension $n-1$ if and only if $f \notin (X_1, \ldots, X_n)^2$.)

Exercise 13.2. Let $\varphi : A \to B$ be an integral ring homomorphism. Show that every prime ideal of A containing $\ker \varphi$ is the preimage of a prime ideal of B. Moreover, show that for every ideal $\mathfrak{b} \subset B$ it holds

$$\dim B/\mathfrak{b} = \dim A/\varphi^{-1}\mathfrak{b}.$$

(*Hint:* Use the Going-up Theorem and Cor. 5.9 from the lectures.)

Exercise 13.3. Let $\varphi : A \to B$ be a homomorphism of rings. Consider the induced map on spectra $\varphi^* : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$, $\mathfrak{P} \longmapsto \varphi^{-1}\mathfrak{P}$. Show that for every $\mathfrak{p} \in \operatorname{Spec} A$ there is a canonical isomorphism of partially ordered sets (ordered by "$\subset$")

$$(\varphi^*)^{-1}(\mathfrak{p}) \xrightarrow{\sim} \operatorname{Spec}\big(B \otimes_A k(\mathfrak{p})\big)$$

between the fiber $F_\mathfrak{p} := (\varphi^*)^{-1}(\mathfrak{p}) = \{\mathfrak{P} \in \operatorname{Spec} B \mid \varphi^{-1}\mathfrak{P} = \mathfrak{p}\}$ of φ^* over $\mathfrak{p}$ and the spectrum of $B \otimes_A k(\mathfrak{p})$, where $k(\mathfrak{p}) = \operatorname{Frac}(A/\mathfrak{p})$ is the residue field at $\mathfrak{p}$.

[1] If you want your solutions of this exercise sheet to be corrected, please hand them in just before the lecture on January 28th. Questions or comments to `henrik.russell@math.fu-berlin.de` or come to office A3, 112.

Exercise 13.4. Let A be a noetherian ring, $A[X]$ the polynomial ring in one indeterminate over A.

(a) Show that for every $\mathfrak{p} \in \operatorname{Spec} A$ it holds

$$\operatorname{height}(\mathfrak{p}[X]) = \operatorname{height}(\mathfrak{p})$$

where $\mathfrak{p}[X]$ is the set of all polynomials in $A[X]$ with coefficients in $\mathfrak{p}$, and this is a prime ideal, see Exc. 8.2.

(*Hint:* Suppose $\operatorname{height}(\mathfrak{p}) = m$. Then there exist $a_1, \dots, a_m \in \mathfrak{p}$ such that $\mathfrak{p}$ is a minimal prime ideal over $\mathfrak{a} = (a_1, \dots, a_m)$. Then $\mathfrak{p}[X]$ is a minimal prime ideal over $\mathfrak{a}[X] = (a_1, \dots, a_m)A[X]$ by Exc. 8.2 e). Now use a corollary of the Dimension Theorem to show that $\operatorname{height}(\mathfrak{p}[X]) \leq m$. On the other hand, a chain of prime ideals in A gives rise to a chain of prime ideals in $A[X]$ of the same length.)

(b) Show

$$\dim A[X] = \dim A + 1.$$

(*Hint:* Let $\mathfrak{P}_0 \subsetneqq \mathfrak{P}_1 \subsetneqq \dots \subsetneqq \mathfrak{P}_n$ be a chain of prime ideals in $A[X]$ of maximal length.

Consider the inclusion $A \overset{\iota}{\subset} A[X]$. The fiber of the associated map on spectra from Exc. 13.3 over any $\mathfrak{p} \in \operatorname{Spec} A$ corresponds to the 1-dimensional ring $k(\mathfrak{p})[X]$. Use this and part (a) to show that $n \geq \dim A + 1$. Show that there is at most one index $0 \leq \nu < n$ such that $\mathfrak{P}_\nu$ and $\mathfrak{P}_{\nu+1}$ lie in the same fiber of $\operatorname{Spec} A[X] \xrightarrow{\iota^*} \operatorname{Spec} A$.)