Jan 14th, 2014

Algebra I

Prof. H. Esnault

Exercise sheet 12[1]

Exercise 1. Let $A := \oplus_{i=0}^{\infty} A_i$ be a graded ring. The following statements are equivalent.

(1) A is noetherian;
(2) A_0 is noetherian and A is a finitely generated A_0-algebra.

Exercise 2. Let $A := \oplus_{i=0}^{\infty} A_i$ be a noetherian graded ring, $M := \oplus_{i=0}^{\infty} M_i$ be a finitely generated graded A-module. Then M_i is finitely generated as an A_0-module for all $i \geq 0$.

Exercise 3. Let $A := \oplus_{i=0}^{\infty} A_i$ be a graded ring, $M := \oplus_{i=0}^{\infty} M_i$ be a graded A-module. Suppose that $N \subseteq M$ is a sub A-module (not necessarily graded). Then the following statements are equivalent.

(1) $N = \oplus_{i=0}^{\infty} (N \cap M_i)$;
(2) N is generated by homogeneous elements of M;
(3) If $x = \oplus_{i=r}^{r+s} x_i \in N$, where $r, s \in \mathbb{N}$, $x_i \in M_i$, then $x_i \in N$.

Exercise 4. Let $P(z) \in \mathbb{Q}[z]$. We call $P(z)$ a numerical polynomial if $P[n] \in \mathbb{Z}$ for all $n \in \mathbb{N}$ sufficiently large. Prove the following statements.

(1) If $P(z)$ is a numerical polynomial, then there are integers $c_0, c_1, \cdots, c_r$ such that
$$P(z) = c_0 \binom{z}{r} + c_1 \binom{z}{r-1} + \cdots + c_r,$$
where $\binom{z}{r} = \frac{1}{r!} z(z-1)\cdots(z-r+1)$ is the binomial coefficient function.

(2) If $f : \mathbb{Z} \to \mathbb{Z}$ is any function, and if there exists a numerical polynomial $Q(z)$ such that the difference function $f(n+1) - f(n)$ is equal to $Q(n)$ for all $n \in \mathbb{N}$ sufficiently large, then there exists a numerical polynomial $P(z)$ such that $f(n) = P(n)$ for all $n \in \mathbb{N}$ sufficiently large.

[1] If you want your solutions to be corrected, please hand them in just before the lecture on Jan.21st. If you have any questions concerning these exercises you can contact Lei Zhang via l.zhang@fu-berlin.de or come to Arnimallee 3 112A.