January 7th, 2014

Algebra I

Prof. H. Esnault

Exercise sheet 11[1]

Exercise 1. Let k be a field and A a finitely generated k-algebra. Prove that the following statements are equivalent:

(i) A is Artinian.
(ii) A is a finite dimensional k-vector space.

(*Hint:* $(i) \Rightarrow (ii)$: Use the structure theorem for Artinian rings to reduce to the case where A is local with maximal ideal $\mathfrak{m}$. Then show by induction over n, that $A/\mathfrak{m}^n$ is a finite dimensional k-vector space, the case $n = 1$ being a special case of the weak Hilbert Nullstellensatz, and conclude. $(ii) \Rightarrow (i)$: Observe that ideals $I \subset A$ are in particular sub-k-vector spaces.)

In the following k is an *algebraically closed field*. Let $A = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables with coefficients in k and $I \subset A$ an ideal. We denote by

$$Z(I) = \{a \in k^n \mid f(a) = 0, \text{ for all } f \in I\}$$

the zero set of I. Notice if $I = (f_1, \ldots, f_r)$, then $Z(I) = \{a \in k^n \mid f_i(a) = 0, \text{ for all } i = 1, \ldots, r\}$. Further we set

$$I(Z(I)) = \{f \in A \mid f(a) = 0 \text{ in } k, \text{ for all } a \in Z(I)\}.$$

Notice that $I(Z(I))$ is an ideal in A, which clearly contains $\sqrt{I}$ the radical of I.

Exercise 2. Let k be an algebraically closed field and $I \subset A := k[x_1, \ldots, x_n]$ an ideal. Show with the notations from above:

(i) We have a bijection

$$Z(I) \xrightarrow{\simeq} \{\text{maximal ideals in } A/I\}, \quad (a_1, \ldots, a_n) \mapsto (\bar{x}_1 - a_1, \ldots, \bar{x}_n - a_n).$$

In particular, $Z(I) \neq \emptyset \Leftrightarrow I \neq (1)$. (*Hint*: Use the weak Hilbert Nullstellensatz to prove the surjectivity.)

(ii)

$$I(Z(I)) = \sqrt{I}.$$

[1]If you want your solutions of this exercise to be corrected, please hand them in just before the lecture on January 14. Questions or comments to kay.ruelling@fu-berlin.de or come to A3, Room 108.

(*Hint*: Take $f \in I(Z(I))$. To show that $f^N \in I$ for some $N \geq 1$ proceed as follows: Show that $A_f \cong A[y]/(1-fy)$ and hence is a finitely generated k-algebra. Show that $IA_f = A_f$, else A_f/IA_f is a finitely generated (non-zero) k-algebra and we can use the weak Hilbert Nullstellensatz to find a maximal ideal $\mathfrak{m} \subset A$ with $I \subset \mathfrak{m}$ and $f \notin \mathfrak{m}$; using (i) we get a contradiction. Now conclude.)

Exercise 3. Let k be an algebraically closed field, $f, g \in k[x, y]$ two polynomials in two variables and set $F = Z((f))$, $G = Z((g))$. Assume that $F \cap G \subset k^2$ is a finite set (possibly empty).

(i) Show that the k-algebra $k[x, y]/(f, g)$ is Artinian.

(ii) For $P = (a, b) \in k^2$ denote by $\mathfrak{m}_P = (x - a, y - b)$ the corresponding maximal ideal of $k[x, y]$. Set

$$I(P, F \cap G) := \dim_k \frac{k[x, y]_{\mathfrak{m}_P}}{(f, g)k[x, y]_{\mathfrak{m}_P}},$$

where $\dim_k$ denotes the dimension as a k-vector space. Show that $I(P, F \cap G)$ is a non-negative integer and that it is not zero iff $P \in F \cap G$.

(iii) Show $\sum_{P \in k^2} I(P, F \cap G) = \dim_k k[x, y]/(f, g)$.

The number $I(P, F \cap G)$ from the exercise above is called the *intersection multiplicity of the plane affine curves F and G at the point P*.

Exercise 4. With the notation from Exercise 3 above, compute $F \cap G$ and $I(P, F \cap G)$ for all $P \in F \cap G$ in the following cases:

(i) $f := y^2 - x^3$, $g := y$.

(ii) $f := (x^2 + y^2)^3 - 4x^2y^2$, $g := x - y$, $\mathrm{char}(k) \neq 2$.

(iii) $f := y - x^2$, $g := y^2 - 2x^3 + x^2$.