October 15th, 2013

Algebra I

Prof. H. Esnault

Exercise sheet 1[1]

Exercise 1. Let A be a ring.

(i) Assume $n \in A$ is a nilpotent element. Show that $1+n$ is a unit of A. (*Hint:* Think about the geometric series trick!)

(ii) Deduce from (i), that if $u \in A^\times$ is a unit and $n \in A$ is nilpotent, then $u+n \in A^\times$ is a unit.

Exercise 2. Let A be a ring and $A[X]$ the polynomial ring in one variable X with coefficients in A. Let $f = a_0 + a_1X + \ldots + a_nX^n \in A[X]$ be a polynomial with $a_i \in A$, $i = 0, \ldots, n$. Prove:

(i) $f \in (A[X])^\times \iff a_0 \in A^\times$ and $a_1, \ldots, a_n$ are nilpotent. (*Hint:* For $\Rightarrow$: If $b_0 + \ldots + b_mX^m$ is an inverse of f show by induction on r that $a_n^{r+1}b_{m-r} = 0$, for $r \geq 0$. Deduce that a_n is nilpotent and use Ex. 1, (ii).)

(ii) f is nilpotent $\iff a_0, a_1, \ldots, a_n$ are nilpotent.

(iii) f is a zero-divisor $\iff \exists\, a \in A \setminus \{0\}$ such that $af = 0$. (*Hint:* For $\Rightarrow$: Choose $g = b_0 + \ldots + b_mX^m \neq 0$ of least degree m such that $fg = 0$. Then $a_nb_m = 0$. Deduce $a_ng = 0$ and by induction $a_{n-r}g = 0$, all $0 \leq r \leq n$. Deduce the statement.)

Describe the units, nilpotent elements and zero-divisors of $A[X]$ in the case where A is an integral domain.

Exercise 3. Which of the following ideals are prime, which are maximal?

(i) (0), (5), (7), (7365) in $\mathbb{Z}$

(ii) (0), (5), (7), (7365) in $\mathbb{Q}$

(iii) $(X-\lambda)$, (X^3), (λ), (X^2-1) in $K[X]$, where K is a field and $\lambda \in K \setminus \{0\}$

(iv) (13), $(2, X^2+X+1)$, $(2, X^3+X^2+X+1)$, (X^2+1) in $\mathbb{Z}[X]$

(v) (πX), (XY), (X^2-Y^2), $(X^2+Y^2)^{(*)}$ in $\mathbb{R}[X,Y]$

(The parts of an exercise which are marked by a $(*)$ may require a little bit more work.)

[1] If you want your solutions of this exercise to be corrected, please hand them in just before the lecture on October 22. Questions or comments to kay.ruelling@fu-berlin.de or come to A3, Room 108.

Exercise 4. Let A be an integral domain and $A[X,Y]$ the polynomial ring in two variables with coefficients in A. Let $m,n \in \mathbb{Z}_{\geq 1}$ be positive integers.

Show that the ideal $(X^m - Y^n)$ is prime in $A[X,Y]$ if and only if m and n are coprime, i.e. $(m,n) = (1)$.
(*Hint:* For the "if" direction: Show that the map $\varphi : A[X,Y] \to A[T]$, $f(X,Y) \mapsto f(T^n, T^m)$ is a ring homomorphism, which factors over a ring homomorphism $\bar{\varphi} : A[X,Y]/(X^m - Y^n) \to A[T]$. Then show that $\bar{\varphi}$ is injective and conclude with Ex. 2.)