Dedekind domains

Definition 1 ([AK2017, 24.1])**.** A ring is a *Dedekind domain* if it is a noetherian normal domain of dimension 1.

Exercise 1 ([AK2017, 24.5])**.** If R is a Dedekind domain, and S is any multiplicative subset, then $R[S^{-1}]$ is also a Dedekind domain if $\mathfrak{p} \cap S = \varnothing$ for some nonzero prime, and otherwise $R[S^{-1}] = \mathrm{Frac}(R)$.

Theorem 2 ([AK2017, 24.6])**.** *Let R be a noetherian domain which is not a field. Then R is a Dedekind domain if and only if $R_{\mathfrak{p}}$ is a dvr for every nonzero prime $\mathfrak{p}$.*

Proof. Let R be a Dedekind domain. Then $R_{\mathfrak{p}}$ is also a Dedekind domain by Exercise 24.5, so $R_{\mathfrak{p}}$ is a dvr by Theorem 23.6. Conversely, suppose that R is a noetherian domain such that $R_{\mathfrak{p}}$ is a dvr for every nonzero prime $\mathfrak{p}$. We must show that R is dimension one and normal. Any maximal chain of prime ideals $(0) = \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_d$ corresponds to a chain in $R_{\mathfrak{p}_d}$. But this is a dvr, so $d = 1$, and we deduce that $\dim R = 1$. Let $a \in K$ be an integral element. Then a is also integral for all $R_{\mathfrak{p}}$. These are all normal, so $a \in R_{\mathfrak{p}}$ for every nonzero $\mathfrak{p}$, and therefore $a \in \cap R_{\mathfrak{p}}$. We claim that $R \subset \cap R_{\mathfrak{p}}$ is an equality. It suffices to show that $R \subset \cap R_{\mathfrak{m}}$ is an equality. Suppose that $a \in \cap R_{\mathfrak{m}}$ is not in R. Then the ideal $I = \{b \in R : ab \in R\}$ is proper, and therefore contained in some maximal ideal $\mathfrak{m}$. But we cannot have $a \in R_{\mathfrak{m}}$, for otherwise, we would have $ca \in R$ for some $c \notin \mathfrak{m} \supset I$. This contradicts the assumption that $a \in \cap_{\mathfrak{m}} R_{\mathfrak{m}}$. □

Theorem 3 ([AK2017, 24.8, 25.14])**.** *Let $I \subset R$ be an ideal of a Dedekind domain I. Then I can be written uniquely as a product $\prod_{i=1}^n \mathfrak{p}_i^{a_i}$ for some set of distinct maximal ideals $\mathfrak{p}_i$ and nonzero positive integers $a_i \in \mathbb{Z}$. In fact, $a_i = \min_{x \in I} v_{\mathfrak{p}_i}(x)$, so we have*

$$I = \prod \mathfrak{p}^{v_{\mathfrak{p}}(I)}$$

if we define $v_{\mathfrak{p}}(I) = \min_{x \in I} v_{\mathfrak{p}}(x)$.

(Recall that since all local rings of R are dvr's, every ideal is locally principle. Since R is noetherian every ideal is finitely generated. So by 25.13, every ideal is invertible).

Example 4. Consider the Dedekind domain $\mathbb{Z}[\sqrt{-5}]$. This is not a UFD because

$$2 \cdot 3 = 6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

but the unit group is just ± 1. On the other hand, there is a unique factorisation

$$(6) = \mathfrak{p}_2^2 \mathfrak{p}_3 \mathfrak{q}_3$$

of the ideal $(6) = \{6a + 6b\sqrt{-5} : a, b \in \mathbb{Z}\}$ into prime ideals. Here,

$$\mathfrak{p}_2 = (2, 1+\sqrt{-5}), \qquad \mathfrak{p}_3 = (3, 1+\sqrt{-5}), \qquad \mathfrak{q}_3 = (3, 1-\sqrt{-5}).$$

Proof. [Eisenbud, Com.Alg with a view toward Alg.Geom., Thm.11.8]. Consider the set of all proper ideals $I \subsetneq R$ which cannot be written in such a fashion and choose a maximal one. Let $\mathfrak{p}$ be a maximal ideal containing I. Since $\mathfrak{p}$ is invertible we have $\mathfrak{p}^{-1}\mathfrak{p} = R$ and so $\mathfrak{p}^{-1} \supsetneq R$. We claim we also have $\mathfrak{p}^{-1} I \supsetneq I$. If $\mathfrak{p}^{-1} I = I$, then I is a finitely generated faithful $R[x]$-module for any $x \in \mathfrak{p}^{-1}$. By [AK2017, 10.14], this implies that x is integral, and therefore in R since Dedekind domains are normal. So $\mathfrak{p}^{-1} I \supsetneq I$. Note $R = \mathfrak{p}^{-1}\mathfrak{p} \supset \mathfrak{p}^{-1} I$. By the assumption of maximality, we can write $\mathfrak{p}^{-1} I = \prod \mathfrak{q}_j$ for some (not necessarily distinct) $\mathfrak{q}_j$. But then $I = \mathfrak{p} \prod \mathfrak{q}_j$.

Now consider a product $I = \prod \mathfrak{p}^{a_\mathfrak{p}}$ over all maximal ideals of R, such that almost all $a_\mathfrak{p}$ zero. Note that any $\mathfrak{q}$ we have $IR_\mathfrak{q} = \mathfrak{q}^{a_\mathfrak{q}} R_\mathfrak{q}$ since $\mathfrak{p} \not\subset \mathfrak{q}$ for all $\mathfrak{p} \neq \mathfrak{q}$ so $\mathfrak{p}R_\mathfrak{q} = R_\mathfrak{q}$. So $a_\mathfrak{q} = \min_{a/s \in IR_\mathfrak{q}} v_\mathfrak{q}(a/s)$. But $v_\mathfrak{q}(a/s) = v_\mathfrak{q}(a)$ since $s \in R^*_{\mathfrak{p}_i}$. □

Corollary 5 ([AK2017, 25.14])**.** *If R is a Dedekind domain, then the group of invertible fractional ideals $F(R)$ is the free abelian group on the set of maximal ideals.*

$$F(R) = \mathbb{Z}\{ \text{ maximal ideals } \}$$

Proof. Note that for any invertible fractional ideal I, we have $xI \subset R$ for some x. Moreover, we have $v_\mathfrak{p}(xI) = v_\mathfrak{p}(x) + v_\mathfrak{p}(I)$. Now $xI = \prod \mathfrak{p}^{v_\mathfrak{p}(xI)}$ and $xR = \prod \mathfrak{p}^{v_\mathfrak{p}(xR)}$ so

$$I = (xI : xR) = \prod \mathfrak{p}^{v_\mathfrak{p}(xI) - v_\mathfrak{p}(xR)} = \prod \mathfrak{p}^{v_\mathfrak{p}(I) - v_\mathfrak{p}(R)} = \prod \mathfrak{p}^{v_\mathfrak{p}(I)}.$$ □