October 17, 2017

Addendum to the course on November 21, 2017

In the sequel, all rings have a unit $1 \neq 0$.

Lemma 1 (AK, Cor. (12.10))**.** *Let R be a ring, $S \subset R$ be a multiplicative subset, M be a module. Then the set theoritic map of R-modules*

$$\mu : S^{-1}R \times M \to S^{-1}M,\ \mu(\frac{x}{s}, m) = \frac{xm}{s} \tag{1}$$

factors through

$$\bar{\mu} : S^{-1}R \otimes_R M \to S^{-1}M \tag{2}$$

and $\bar{\mu}$ is an isomorphism of $S^{-1}R$-modules.

We recall that the $S^{-1}R$-module structure on $S^{-1}R \otimes_R M$ is defined by

$$\frac{y}{t}(\frac{x}{s} \otimes m) = \frac{t}{t}\frac{x}{s} \otimes m. \tag{3}$$

Proof. The map μ factors through $\bar{\mu}$ if and only if μ is R-bilinear. This is trivially checked. We want to prove that $\bar{\mu}$ is bijective. Since by definition any element in $S^{-1}M$ is of the shape $\frac{m}{s} = \bar{\mu}(\frac{1}{s} \otimes m)$ for some $m \in M$ and $s \in S$, $\bar{\mu}$ is surjective. We now prove injectivity. Recall:

Claim 2. Any element of $S^{-1}R \otimes_R M$ is of the shape $\frac{1}{s}(1 \otimes m)$ for some $s \in S$ and $m \in M$.

Proof. Any element of $S^{-1}R \otimes_R M$ is by definition of the shape

$$\sum_{i=1}^{n} \frac{x_i}{s_i} \otimes m_i$$

for some $x_i \in R, s_i \in S, m_i \in M$. Write $\frac{x_i}{s_i} = \frac{y_i}{s}$ for some $y_i \in R, s \in S$ (one can give explicit values for y_i and s, but this is irrelevant for our purpose). Thus

$$\sum_{i=1}^{n} \frac{x_i}{s_i} \otimes m_i = \sum_{i=1}^{n} \frac{1}{s}(y_i \otimes m_i) = \sum_{i=1}^{n} \frac{1}{s}(1 \otimes y_i m_i) =$$

$$\frac{1}{s}(1 \otimes m) \text{ where } m = \sum_{i=1}^{n} y_i m_i.$$

□

End of Proof of the Lemma 1. If $\bar{\mu}(\frac{1}{s}(1 \otimes m)) = \frac{m}{s} = 0$ then there is $\sigma \in S$ such that $\sigma m = 0$ thus $\frac{1}{s}(1 \otimes m) = \frac{\sigma}{\sigma s}(1 \otimes m) = \frac{1}{\sigma s}(\sigma \otimes m) = \frac{1}{\sigma s}(1 \otimes \sigma m) = \frac{1}{\sigma s} \otimes 0 = 0$. Thus $\bar{\mu}$ is injective. This finishes the proof. □

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