October 23, 2017

Number Theory I (Commutative Algebra)

Hélène Esnault, Exercises: Shane Kelly

Exercise sheet 1

As in the textbook, all rings are commutative with a unit.
Recall that an element of a ring $e \in R$ is an *idempotent* if $e^2 = e$. An element $a \in R$ is *nilpotent* if $a^n = 0$ for some $n \geq 1$. An element $a \in R$ of a unital ring is *invertible* if there is some $b \in R$ with $ab = 1$.

Exercise 1. Given an integer $n > 1$, find all nilpotents in $\mathbb{Z}/\langle n \rangle$. Find all units.

Exercise 2 ([AK2012, Exercise.1.14])**.** First, given a prime number p and a $k \geq 1$, find the idempotents in $\mathbb{Z}/\langle p^k \rangle$. Second, find the idempotents in $\mathbb{Z}/\langle 12 \rangle$. Third, find the number of idempotents in $\mathbb{Z}/\langle n \rangle$ where $n = \prod_{i=1}^{N} p_i^{n_i}$ with p_i distinct prime numbers.

Exercise 3 ([AK2017, Exercise.1.21])**.** (Chinese Remainder Theorem). Let R be a ring.

(1) Let $\mathfrak{a}$ and $\mathfrak{b}$ be comaximal ideals; that is, $\mathfrak{a} + \mathfrak{b} = R$. Sometimes this is called being *coprime*. Prove that $\mathfrak{a}\mathfrak{b} = \mathfrak{a} \cap \mathfrak{b}$ and $R/\mathfrak{a}\mathfrak{b} = (R/\mathfrak{a}) \times (R/\mathfrak{b})$.
(2) Given $m, n \geq 1$, show $\mathfrak{a}$ and $\mathfrak{b}$ are comaximal if and only if $\mathfrak{a}^m$ and $\mathfrak{b}^n$ are.
(3) Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be pairwise comaximal. Prove
 (a) $\mathfrak{a}_1$ and $\mathfrak{a}_2 \dots \mathfrak{a}_n$ are comaximal;
 (b) $\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n = \mathfrak{a}_1 \cdots \mathfrak{a}_n$;
 (c) $R/(\mathfrak{a}_1 \cdots \mathfrak{a}_n) \xrightarrow{\sim} \prod (R/\mathfrak{a}_i)$.
(4) Find an example where $\mathfrak{a}$ and $\mathfrak{b}$ satisfy (1)(a), but aren't comaximal.

Exercise 4 ([AK2017, Exercise.1.24])**.** Let R be a ring, and e, e' idempotents.

(1) Let $\mathfrak{a} = \langle e \rangle$. Show $\mathfrak{a}$ is idempotent; that is, $\mathfrak{a}^2 = \mathfrak{a}$.
(2) Let $\mathfrak{a}$ be a principal idempotent ideal. Show $\mathfrak{a} = \langle f \rangle$ with f idempotent.
 Assume $\langle e \rangle = \langle e' \rangle$. Show $e = e'$.
(3) Set $e'' = e + e' - ee'$. Show $\langle e, e' \rangle = \langle e'' \rangle$ and e'' is idempotent.
(4) Let $e_1, \dots, e_r$ be idempotents. Show $\langle e_1, \dots, e_r \rangle = \langle f \rangle$ with f idempotent.
(5) Assume R is Boolean, that is, $a^2 = a$ for every $a \in R$. Show every finitely generated ideal is principal.