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# Remark on proposition 4.1 (b) of Milne's book

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In the proof of pProposition 4.1 (b) there is a statement which is not proven, I will prove the statement here.

**Proposition 1.** *Let $A$ be a Dedekind domain with field of fractions $K$ and let $B$ be the integral closure of $A$ in a finite separable extension $L/K$. We denote by* $\mathrm{Id}(A)$*, resp.* $\mathrm{Id}(B)$*, the group of fractional ideals of $A$, resp. of $B$. The map*

$$\mathrm{Id}(A) \longrightarrow \mathrm{Id}(B)$$
$$I \longmapsto IB$$

*is injective.*

*Proof.* Let $I \subseteq A$ be an ideal and $I \neq A$, then there exists a maximal ideal $P$ such that $I \subseteq P$. Since the ring extension $A \subseteq B$ is integral, there exists a prime ideal $Q$ of $P$ such that $Q \cap A = P$ [AM69, Theorem 5.10]. Then $IB \subseteq (Q \cap A)B \subseteq QB = Q$, hence $IB \neq B$.

We take now a fractional ideal $J$ of $A$. We can write it as $J = LN$, where $L \subseteq A$ and $N^{-1} \subseteq A$ and we can suppose that the primes in the factorization of $L$ and $N$ are different. Hence if $JB = B = LBNB$, thins means that $LB = (NB)^{-1}$. Suppose that there exists a prime ideal $Q$ of $B$ such that $Q \mid LB$ and $Q \mid N^{-1}B$, then if we denote by $P$ the prime ideal of $A$ given by $Q \cap A$, we have that $L \subseteq LB \subseteq Q$, hence $L \subseteq Q$, and since $L \subseteq A$, hence $L \subseteq Q \cap A = P$ and the same is true for $N^{-1}$, i.e. $N^{-1} \subseteq P$. But we supposed that the primes dividing $N^{-1}$ and $L$ were different, so we got an absurd. Hence looking at the prime factorization we conclude that $LB = B$ and $(NB)^{-1} = B$. By what we have proven before we conclude that $L = A$ and $N^{-1} = A$. $\square$

## References

[AM69] M. F Atiyah and I. G Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., 1969.