May 17th, 2016

Number Theory II

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Exercise sheet 5[1]

Remark 1. We assume, to solve this exercise sheet, that the ring of integers of a number field is a Dedekind domain (Theorem 3.29 of Milne's book).

Exercise 1. Let d be a square free integer and p a prime number not dividing $2d$. Let $\mathcal{O}_K$ be the ring of integer of $\mathbb{Q}(\sqrt{d})$. Show that $p\mathcal{O}_K$ is a prime ideal if and only if the congruence $x^2 \equiv d \pmod{p}$ has no solution.

Exercise 2. We consider the ring $R = \mathbb{Z}[\sqrt{-5}]$.

(i) Show that $2R = (2, 1 + \sqrt{-5})^2$.

(ii) Find the unique factorization of $6R$ into prime ideals.

Exercise 3. Prove that for every $d \equiv 1 \pmod{4}$ such that $\frac{d+1}{2}$ is a prime number, the ring $\mathbb{Z}[\sqrt{-d}]$ is not UFD and that its ideal class group has an element of order 2.

Exercise 4. Let K be a number field with ring of integers $\mathcal{O}_K$ and let $I \subset \mathcal{O}_K$ be an ideal. Show that the finiteness of the class group of $\mathcal{O}_K$ implies that there exists a finite extension L/K such that $I\mathcal{O}_L$ is principal, where $\mathcal{O}_L$ is the ring of integers of L.

Exercise 5. Let R be a Dedekind domain and let $I \neq 0$ be an ideal of R; prove that every ideal in R/I is principal.

[1] If you want your solutions of this exercises to be corrected, please hand them in before the exercise class on May 27th.