ZAHLENTHEORIE II – ÜBUNGSBLATT 6

PROF. DR. HÉLÈNE ESNAULT AND DR. LEI ZHANG

Exercise 1. Let $K = \mathbb{Q}[i]$.

(a) Compute the discriminant of $K/\mathbb{Q}$.
(b) Show that $\mathbb{Z}[i]$ is the ring of integers of K.
(c) Use the big theorem we introduced in the class to prove that $\mathbb{Z}[i]$ is a principal ideal domain.

Exercise 2. Let $K = \mathbb{Q}[\sqrt{-5}]$.

(a) Compute the discriminant of $K/\mathbb{Q}$.
(b) Show that $\mathbb{Z}[\sqrt{-5}]$ is the ring of integers of K.
(c) Compute the group $\mathrm{Cl}(\mathbb{Z}[\sqrt{-5}])$.

Exercise 3. Let K be a number field, and let $I \subseteq \mathcal{O}_K$ be an ideal.

(a) Use the fact that $\mathrm{Cl}(\mathcal{O}_K)$ is finite to show that there is a finite extension L/K such that $I\mathcal{O}_L$ is a principal ideal. (Hint: There exists $n \in \mathbb{N}$ such that $I^n = (a)$ for some $a \in \mathcal{O}_K$. Now consider $a^{\frac{1}{n}}$. Show that $I = (a^{\frac{1}{n}})$.)
(b) Show that there is a finite extension L/K such that for any ideal $J \subseteq \mathcal{O}_K$ the extension $J\mathcal{O}_L$ is a principal ideal.

Exercise 4. We used in the class the following fact: Let A be a commutative ring, and let $p, p_1, p_2, \cdots, p_n$ be prime ideals of A. If $p_1 \cap p_2 \cap \cdots \cap p_n \subseteq p$ then $p_i \subseteq p$ for some $1 \leq i \leq n$. Please show this fact as an exercise.

If you want your solutions to be corrected, please hand them in just before the lecture on May 30, 2017. If you have any questions concerning these exercises you can contact Dr. Lei Zhang via l.zhang@fu-berlin.de or come to Arnimallee 3 112A.