ZAHLENTHEORIE II – ÜBUNGSBLATT 3

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Exercise 1. In this exercise we would like to give an example of a commutative ring A all of whose local rings are Noetherian but itself is not.

(a) Recall that a commutative ring A is called *von Neumann* if for all $a \in A$ there exists $s \in A$ with the equation $a = sa^2$. Show that a field is von Neumann, and show also that product of von Neumann rings and quotient of a von Neumann ring are von Neumann rings.

(b) Show that if A is both a domain and a von Neumann ring, then A is a field.

(c) Show that if A is a von Neumann ring, then A is 0-dimensional, i.e. any prime ideal maximal.

(d) Show that if A is a reduced (i.e. there exists no non-zero element $a \in A$ with the property that $a^n = 0$ for some $n \in \mathbb{N}$) and 0-dimensional ring, then for any prime ideal $\mathfrak{p} \subseteq A$ the local ring $A_\mathfrak{p}$ is a field.

(e) Let $A := \prod_{s \in S} K_s$, where S is an infinite set and K_s is a field for every $s \in S$. Show that for any prime ideal $\mathfrak{p} \subseteq A$ the local ring $A_\mathfrak{p}$ is a field, and show also that A is not Noetherian.

Exercise 2. Let A be a commutative ring.

(a) Suppose that for any maximal ideal $\mathfrak{m}$ the local ring $A_\mathfrak{m}$ is Noetherian and that for any non-zero element $a \in A$ there are only finitely many maximal ideals of A which contain a. Show that A is Noetherian. (Hint: For any ascending chain $0 \neq I_0 \subseteq I_1 \subseteq \cdots$ there are only finitely many maximal ideals which contain I_0. Thus any such chain stabilizes in all $A_\mathfrak{m}$.)

(b) Suppose that A is an integral domain in which any non-zero element is contained in only finitely many maximal ideals and that for any maximal ideal $\mathfrak{m}$ the local ring $A_\mathfrak{m}$ is a DVR. Show that A is a Dedekind domain.

Exercise 3. In this exercise we would like to compute the ring of integers of a quadratic extension by hand. Let $R := \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$, and let $\alpha = r + s\sqrt{5}$ with $r, s \in \mathbb{Q}$.

(a) Show that the minimal polynomial of α is $\phi(X) = X^2 - 2rX + r^2 - 5s^2$.

(b) Show that α is integral over $\mathbb{Z}$ if and only if $\phi(X) \in \mathbb{Z}[X]$.

(c) Show that if α is integral over $\mathbb{Z}$ and if $r \in \mathbb{Z}$ then $\alpha \in \mathbb{Z}[\sqrt{5}] \subseteq R$.

(d) Show that if α is integral over $\mathbb{Z}$ and if $r \notin \mathbb{Z}$ then $\alpha - \frac{1+\sqrt{5}}{2} \in \mathbb{Z}[\sqrt{5}] \subseteq R$.

(e) Conclude that $R := \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ is integrally closed.

(f) Conclude that R is the ring of integers of $\mathbb{Q}[\sqrt{5}]$.

If you want your solutions to be corrected, please hand them in just before the lecture on May 9, 2017. If you have any questions concerning these exercises you can contact Dr. Lei Zhang via `l.zhang@fu-berlin.de` or come to Arnimallee 3 112A.