ZAHLENTHEORIE II – ÜBUNGSBLATT 12

PROF. DR. HÉLÈNE ESNAULT AND DR. LEI ZHANG

Exercise 1. Let A be a complete discrete valuation ring with fraction field K. We know that K has a complete absolute value defined by A. Show that if V is a finite dimensional K vector space, then V has a complete norm. In fact any two norms $|\cdot|_1$ and $|\cdot|_2$ on V are equivalent, that is, there exist two positive real numbers α, β such that $\alpha|x|_2 \leq |x|_1 \leq \beta|x|_2$. Using this fact to show that if L is a finite extension of K, then there is a unique absolute value on L extending the one on K.

Exercise 2. Let A be a complete discrete valuation ring with fraction field K. Let L be a finite separable extension of K. Using the previous exercise to show that there exists a unique absolute value on L extending the one on K. (Hint: In view of the previous exercise we only have to show the existence. First of all we know from the class that the integral closure B of A in L is Dedekind. If the integral closure has two maximal ideals, then they will provide different valuations. Another way of seeing this is that since A is Henselian, B is a product of the localizaitons all of its maximal ideals. Thus B has to be local. That is B is a DVR. Thus B provides a discrete valuation on L.)

Exercise 3. Show that there is a unique absolute value on $\bar{\mathbb{Q}}_p$ extending the p-adic absolute value on $\mathbb{Q}_p$.

Exercise 4. Let A be a complete DVR whose fraction field is K. Let L/K be a finite separable field extension. Show that the maximal ideal $\mathfrak{m}$ of A is unramified if and only if $\mathfrak{m}\mathcal{O}_L$ is a prime ideal in $\mathcal{O}_L$, and this is also equivalent to the condition that $[\mathcal{O}_L/\mathfrak{m}_L : A/\mathfrak{m}]$ is equal to the degree of the extension L/K, where $\mathfrak{m}_L$ is the maximal ideal of $\mathcal{O}_L$. If K satisfies one of the equivalent properties, then K is called unramified. Show that there is a maximal unramified extension inside $\bar{K}$, i.e. an extension which contains all the unramified extensions and in which every finite extension is unramified.

Exercise 5. Let A be a complete local ring with maximal ideal $\mathfrak{m}$ and residue field $\kappa := A/\mathfrak{m}$. Let $f(X) \in A[X]$ be a monic polynomial. Show that if $\bar{f}(X) = \bar{g}(X)\bar{h}(X)$, and if $\bar{g}(X), \bar{h}(X)$ are mutually coprime monic polynomials, then there exist monic $g(X), h(X) \in A[X]$ whose reduction mod $\mathfrak{m}$ are $\bar{g}(X), \bar{h}(X)$ respectively.

Exercise 6. Show that inside $\bar{\mathbb{Q}}_p$ there is exactly one unramified extension of $\mathbb{Q}_p$ of a given degree $f \geq 1$. This extension is obtained by adjoin a $p^f - 1$ primitive root of unity of $\bar{\mathbb{Q}}_p$. If

If you want your solutions to be corrected, please hand them in just before the lecture on Juli 11, 2017. If you have any questions concerning these exercises you can contact Dr. Lei Zhang via `l.zhang@fu-berlin.de` or come to Arnimallee 3 112A.

E is the maximal unramified extension of $\mathbb{Q}_p$ inside $\bar{\mathbb{Q}}_p$, then $\mathcal{O}_E/\mathfrak{m}_E = \bar{\mathbb{F}}_p$, where $\mathcal{O}_E$ is the valuation ring and $\mathfrak{m}_E$ is its maximal ideal. For each finite subextension $F \subseteq E$, we can choose a set of representatives $S_F \subseteq \mathcal{O}_F$ such that $S_F \to \mathcal{O}_F \to \mathcal{O}_F/\mathfrak{m}_F$ is an isomorphism of sets. Show that any element x in F is written uniquely as

$$x = \frac{a_{-n}}{p^n} + \frac{a_{-n+1}}{p^{n-1}} + \cdots + a_0 + a_1 p + \cdots$$

where $a_i \in S_F$, $a_{-n} \neq 0$.

Exercise 7. Show that $\bar{\mathbb{Q}}_p$ is not complete.