ZAHLENTHEORIE II – ÜBUNGSBLATT 1

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Exercise 1. Let L/K be a finite field extension. Recall that the extension can always be decomposed as a composition $K \subseteq E \subseteq L$ of field extensions where E/K is finite separable and L/E is finite purely inseparable. If the characteristic of K is 0, then $L = E$, and if the characteristic is $p > 0$, then the degree $[L : E]$ of the extension L/E is p^n for some natural number $n \in \mathbb{N}$. We will use $[L : K]_i$ to denote $[L : E]$ and call it the purely inseparable degree of L/K. Prove the following statements.

(a) If $\alpha \in L$, then $\mathrm{Nm}_{L/K}(\alpha) = (\mathrm{Nm}_{K(\alpha)/K}(\alpha))^{[L:K(\alpha)]}$.

(b) If $\alpha \in L$, then $\mathrm{Tr}_{L/K}(\alpha) = [L : K(\alpha)] \,\mathrm{Tr}_{K(\alpha)/K}(\alpha)$.

(c) If $\{\sigma_i\}_{1\leq i\leq r}$ are the distinct embeddings from L into the algebraic closure $\bar{K}$ of K fixing K, then

$$\mathrm{Nm}_{L/K}(\alpha) = (\prod_{i=1}^{r} \sigma_i(\alpha))^{[L:K]_i}$$

for all $\alpha \in L$.

(d) Notations being as above, we have

$$\mathrm{Tr}_{L/K}(\alpha) = [L : K]_i(\sum_{i=1}^{r} \sigma_i(\alpha))$$

for all $\alpha \in L$.

(e) If $[L : K]_i \neq 1$, then $\mathrm{disc}(L/K) = 0$.

Exercise 2. Here is a small application of norm.

(a) Compute the norm of $1 + \sqrt[3]{2}$ for the field extension $\mathbb{Q}[\sqrt[3]{2}]/\mathbb{Q}$.

(b) Use the computation in (a) to show that there is no element $\alpha \in \mathbb{Q}[\sqrt[3]{2}]$ such that $\alpha^2 = 1 + \sqrt[3]{2}$.

Exercise 3. Let L/K be a finite Galois extension. Let $A \subseteq L$ a subring. Suppose that for any $\sigma \in \mathrm{Gal}(L/K)$ we have $\sigma(A) = A$. Let $R := A \bigcap K$. Show the following statements.

(a) The ring A is integral over R.

(b) If A is integrally closed then so is R.

Exercise 4. Let $A \subseteq B$ be an integral extension of rings. If P is a prime ideal of B, then P is maximal if and only if $P \bigcap A$ is a maximal ideal in A.

If you want your solutions to be corrected, please hand them in just before the lecture on April 25, 2017. If you have any questions concerning these exercises you can contact Dr. Lei Zhang via `l.zhang@fu-berlin.de` or come to Arnimallee 3 112A.