ZAHLENTHEORIE II – ÜBUNGSBLATT 9

PROF. DR. HÉLÈNE ESNAULT AND DR. LEI ZHANG

Exercise 1. Let $|\cdot|_1$, $|\cdot|_2$ be absolute values on K, with $|\cdot|_1$ nontrivial.

(a) Show that if $|\cdot|_1$, $|\cdot|_2$ define the same topology on K then $|\alpha|_1 < 1$ for all $\alpha \in K$ implies that $|\alpha|_2 < 1$ for all $\alpha \in K$.
(b) Show that if $|\alpha|_1 < 1$ for all $\alpha \in K$ implies that $|\alpha|_2 < 1$ for all $\alpha \in K$ and if $b \in \mathbb{Q}$, then $|x|_1 = |y|_1^b$ implies that $|x|_2 = |y|_2^b$.
(c) Show that if $|\alpha|_1 < 1$ for all $\alpha \in K$ implies that $|\alpha|_2 < 1$ for all $\alpha \in K$ and if $b \in \mathbb{R}$, then $|x|_1 = |y|_1^b$ implies that $|x|_2 = |y|_2^b$.
(d) Show that if $|\alpha|_1 < 1$ for all $\alpha \in K$ implies that $|\alpha|_2 < 1$ for all $\alpha \in K$, then $|\cdot|_1 = |\cdot|_2^a$ for some $a > 0$.
(e) Show that if $|\cdot|_1 = |\cdot|_2^a$ for some $a > 0$, then $|\cdot|_1$ and $|\cdot|_2$ define the same topology on K.

Exercise 2. Let $p \in \mathbb{Z}$ be a prime number. Let

$$\mathbb{Z}_p \coloneqq \varprojlim_{n \in \mathbb{N}^+} \mathbb{Z}/p^n\mathbb{Z}$$

and let $\mathbb{Q}_p$ be its fraction field.

(a) Show that any element of $\mathbb{Z}_p$ is uniquely written as an infinite sum

$$a_0 + a_1 p + a_2 p^2 + \cdots$$

where $0 \leq a_i < p$.
(b) Show that $a \in \mathbb{Z}_p$ is invertible iff $a_0 \neq 0$ in the unique expression.
(c) Show that any element of $\mathbb{Z}_p$ is uniquely written as an infinite sum

$$a_{-n}\frac{1}{p^n} + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots$$

with $a_{-n} \neq 0$.
(d) Show that if we assign to each $a = a_n p^n + \cdots \in \mathbb{Q}_p$ with $a_n \neq 0$ the value p^{-n}, then we get a non-archimedean absolute value on $\mathbb{Q}_p$.
(e) Show that

$$a_{-n}\frac{1}{p^n} + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots$$

is the limit of the Cauchy sequence $b_i \coloneqq a_{-n}\frac{1}{p^n} + \cdots + a_i p^i$ under the absolute value.
(f) Show that there is a natural map $\mathbb{Q} \to \mathbb{Q}_p$ which is the completion map with respect to the p-adic valuation on $\mathbb{Q}$.

If you want your solutions to be corrected, please hand them in just before the lecture on Juli 03, 2018. If you have any questions concerning these exercises you can contact Dr. Lei Zhang via `l.zhang@fu-berlin.de` or come to Arnimallee 3 112A.

Exercise 3. We now finish the proof of Hensel's lemma. Let A be a complete local ring with maximal ideal $\mathfrak{m}$. Let $f(X)$ be a degree m polynomial whose reduction mod $\mathfrak{m}$ split as a product of two coprime monic polynomials $\bar{g}(X)$ and $\bar{h}(X)$ in $A/\mathfrak{m}[X]$. Remember that we have to construct $g_n(X), h_n(X) \in A[X]$ of degree r and $m-r$ respectively such that

(a) $g_1(X) = \bar{g}(X)$ and $h_1(X) = \bar{h}(X)$.
(b) $g_{n+1}(X) \equiv g_n(X) \pmod{\mathfrak{m}^n}$
(c) $h_{n+1}(X) \equiv h_n(X) \pmod{\mathfrak{m}^n}$
(d) $f(X) \equiv h_n(X)g_n(X) \pmod{\mathfrak{m}^n}$

We use induction on $n \in \mathbb{N}^+$.

(a) Show that when $n = 1$ we get what we want.
(b) From now on we suppose that we are given $g_n(X)$ and $h_n(X)$. We set $f_n(X) = f(X) - g_n(X)h_n(X) \in \mathfrak{m}^n[X]$. Show that there is $\alpha(X), \beta(X) \in A[X]$ such that
$$g_n(X)\alpha(X)f_n(X) + h_n(X)\beta(X)f_n(X) \equiv f_n(X) \pmod{\mathfrak{m}^{n+1}}$$
(c) Using Euclidean algorithm we get
$$\beta(X)f_n(X) = q(X)g_n(X) + p_n(X)$$
with $p_n(X)$ a polynomial of degree $< r$. Show that $q(X) \in \mathfrak{m}^n[X]$ and so is $p_n(X)$.
(d) Deduce from the formula
$$g_n(X)(h_n(X)q(X) + \alpha(X)f_n(X)) + h_n(X)p_n(X) \equiv f_n(X) \pmod{\mathfrak{m}^{n+1}}$$
that there exists a polynomial $q_n(X) \in \mathfrak{m}^n[X]$ of degree $< m - r$ satisfying
$$g_n(X)q_n(X) + h_n(X)p_n(X) \equiv f_n(X) \pmod{\mathfrak{m}^{n+1}}$$
(e) Show that $g_{n+1}(X) := g_n(X) + p_n(X)$ and $h_{n+1}(X) := h_n(X) + q_n(X)$. This pair will do the job.