ZAHLENTHEORIE II – ÜBUNGSBLATT 8

PROF. DR. HÉLÈNE ESNAULT AND DR. LEI ZHANG

Exercise 1. Let ζ_n be a primitive n-th root of 1, $n \geq 2$. In the class we have shown that if p ramifies then $p|n$. Is the converse true?

Exercise 2.

(a) Recall for $m > 1$ that $\phi(m)$ denotes the order of the group $(\mathbb{Z}/m\mathbb{Z})^*$. Show that if $m, n > 1$ are coprime to each other, then $\phi(mn) = \phi(m)\phi(n)$.

(b) Let $m, n > 1$ be natural numbers. Show that $\phi(mn) \geq \phi(n)$, and the equality holds if and only if $m = 2$ and n is odd.

(c) Let $n > 1$ be an integer. Show that $\mu(\mathbb{Q}(\zeta_n))$ has order n when n is even and has order $2n$ when n is odd.

Exercise 3. Let m, n be mutually coprime natural numbers which are greater than 1. Show that $\mathbb{Q}(\zeta_m) \bigcap \mathbb{Q}(\zeta_n) = \mathbb{Q}$.

Exercise 4. Let K be a number field.

(a) We say that a number field is *cyclotomic* if it is of the shape $\mathbb{Q}(\zeta)$ for some root of unity ζ. Show that every number field K contains a maximal cyclotomic subextension, i.e. a subextension $K_c \subseteq K$ such that every cyclotomic subextension of K is contained in K_c.

(b) Assume that $[K : \mathbb{Q}] = 4$. What are the possibilities for K_c?

If you want your solutions to be corrected, please hand them in just before the lecture on Juni 26, 2018. If you have any questions concerning these exercises you can contact Dr. Lei Zhang via `l.zhang@fu-berlin.de` or come to Arnimallee 3 112A.