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Algebraic Groups

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Exercise sheet 9[1]

Exercise 1. (1) Let Mo_n of GL_n be the subgroup functor consisting of matrices which have exactly one non-zero element in each column and each row;

(2) Show that Mo_n contains both the diagonal matrices $\mathbb{D}_n$ and the permutation matrices S_n, i.e. matrices with exactly one entry of 1 in each column and row.

(3) Show also that S_n is isomorphic to the constant permutation group schemes;

(4) Show that there is an exact sequence

$$1 \to \mathbb{D}_n \to \mathrm{Mo}_n \to S_n \to 1$$

(5) Show that the above sequence splits so that $\mathrm{Mo}_n = \mathbb{D}_n \rtimes S_n$ is representable by a scheme;

(6) Show that the above sequence is the connected-étale sequence of the group scheme Mo_n;

(7) Conclude that Mo_2 is solvable and but not connected;

(8) Show that Mo_2 is not trigonalizable.

Exercise 2. Let k be a field of characteristic 2.

(1) Let G be the kernel of the special linear algebraic group under the relative Frobenius morphism:

$$\mathrm{SL}_2 \xrightarrow{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a^2 & b^2 \\ c^2 & d^2 \end{bmatrix}} \mathrm{SL}_2$$

Show that G is finite connected but not smooth.

(2) Show that there is an exact sequence

$$1 \to \mu_2 \xrightarrow{a \mapsto \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}} G \xrightarrow{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (ab, cd)} \alpha_2 \times \alpha_2 \to 1$$

[1] If you want your solutions to be corrected, please hand them in just before the lecture on June 22, 2016. If you have any questions concerning these exercises you can contact Lei Zhang via l.zhang@fu-berlin.de or come to Arnimallee 3 112A.

(3) Show that G is not trigonalizable. (Hint: You don't have to resort to brutal computation. Just note that if G was trigonalizable then the above exact sequence would tell us that $G = \mu_2 \times \alpha_2 \times \alpha_2$.)

Exercise 3. Show that there is a subgroup scheme of SL_2 consisting of matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ and show that this subgroup scheme is connected smooth but not trigonalizable if the base field is $\mathbb{R}$.

Exercise 4. Let G be a smooth connected affine group scheme over a field k.

(1) Show that the ideals $I_n \subseteq k[G]$ of the map $G^{2n} \subseteq G$ defined in the class are equal to the ideal I of the closed embedding $\mathcal{D}G \subseteq G$ for n sufficiently large;
(2) Let V_n be the image of $G^{2n} \to G$ and let $W_n(k^s) \subseteq G(k^s)$ be the image of $G^{2n}(k^s)$ in $G(k^s)$. Then $|V_n|$ is the closure of $|W_n(k^s)|$;
(3) Show that there is an open dense subset $U \subseteq V_n$ such that $U(k^s) \subseteq W_n(k^s)$; (Hint: $G^{2n} \to G$ is generically flat.)
(4) Suppose n is a number such that $I_n = I$. Show that $U(k^s) \cdot U^{-1}(k^s) \subseteq \mathcal{D}(G(k^s)) \subseteq (\mathcal{D}G)(k^s)$;
(5) Using (4) and the fact that U is open dense to show that $U(k^s) \cdot U^{-1}(k^s) = \mathcal{D}G(k^s)$;
(6) Conclude that $\mathcal{D}(G(k^s)) = (\mathcal{D}G)(k^s)$.