November 27, 2018

Addendum to the course on November 27, 2018

Dear students,

here an addendum to [Mil13, II, Thm.3.10]. Milne's proof contains a few typos and the main points are not really underlined, so we go through the proof of the statement.

Theorem 1. *Let G be a finite group. Let M be a G-module such that $H^1_T(H,M) = H^2_T(H,M) = 0$ for all subgroups $H \subset G$. Then $H^r_T(G,M) = 0$ for all $r \in \mathbb{Z}$.*

In the proof below, we constantly use that by definition $H^r_T(G,M) = H^r(G,M)$ for $r \geq 1$, we do not repeat this point each time.

Proof. When G is cyclic, one applies [Mil13, Prop.3.4].

Assume next G is solvable, i.e. there is a sequence of subgroups $\ldots \subset G_{i+1} \subset G_i \subset \ldots \subset G = G_0$ such that for all $i \in \mathbb{Z}$ we have that $G_{i+1} \neq G_i$ is normal and the quotient group G_i/G_{i+1} is abelian. In particular G_0/G_1 is abelian, and as such, has a cyclic quotient $G_0/G_1 \twoheadrightarrow C \neq 0$, which is a quotient $G_0 = G \twoheadrightarrow C$ of G by precomposing with the quotient homomorphism $G_0 \twoheadrightarrow G_0/G_1$. We set $H = \mathrm{Ker}(G \twoheadrightarrow C)$. By [Mil13, Prop.1.34] one has an exact sequence

$$(1) \qquad 0 \to H^r_T(C, M^H) \to H^r_T(G,M) \to H^r_T(H,M) \ \forall r \geq 1$$

Thus

$$(2) \qquad H^1_T(C, M^H) = H^2_T(C, M^H) = 0$$

and by [Mil13, Prop.3.4], since C is cyclic, one has

$$(3) \qquad H^r_T(C, M^H) = 0 \ \forall r \in \mathbb{Z}.$$

Since M viewed as a H-module verifies the assumption of the theorem and $|H| < |G|$, one has by induction on $|G|$ that

$$(4) \qquad H^r_T(H, M) = 0 \ \forall r \in \mathbb{Z}.$$

Thus by (1) one obtains

$$(5) \qquad H^r_T(G, M) = 0 \ \forall r \geq 1.$$

The next point is to show that $H^0_T(G,M) = 0$, which is to say that any element in $x \in M^G$ is a norm $x = \mathrm{Nm}_G(z)$ for a certain $z \in M$. One has $x \in M^G \subset M^H$ thus x induces a class in $H^0_T(C, M^H) = M^H/\mathrm{Nm}_C M^H$, and this latter group is 0 by (3). Thus there is $y \in M^H$ such that $x = \mathrm{Nm}_C y$. By (4) applied to $r = 0$ one has $H^0_T(H,M) = M^H/\mathrm{Nm}_H M = 0$ thus there is $z \in M$ such that $y = \mathrm{Nm}_H(z)$. Thus one obtains

$$(6) \qquad x = \mathrm{Nm}_C \mathrm{Nm}_H(z) = \mathrm{Nm}_G(z) \text{ thus } H^0_T(G,M) = 0.$$

Hence bringing (5) and (6) together reads

$$H^r_T(G,M)=0\ \forall r\geq 0. \tag{7}$$

We now wish to show that $H^r_T(G,M)=0$ for $r<0$. To this aim we use the standard method for shifting the cohomological degree. One defines the G-module M' by the exact sequence

$$0\to M'\to \mathbb{Z}[G]\otimes_{\mathbb{Z}} M \xrightarrow{\text{valuation}} M\to 0 \tag{8}$$

where the evaluation map assignes to $(g\otimes m)\in \mathbb{Z}[G]\otimes_{\mathbb{Z}} M$ the element $g\cdot m\in M$. On the other hand one has an extension of Shapiro's lemma in [Mil13, Prop.3.1]

Claim 2. For G a finite group and $N=\mathbb{Z}[G]\otimes_{\mathbb{Z}} M$ an induced G-module, one has $H^r_T(G,N)=0$ for all $r\in\mathbb{Z}$.

Granted this general form of Shapiro's lemma, we conclude from the long exact sequence in Tate cohomology associated to (8) that

$$H^r_T(G,M)=H^{r+1}_T(G,M')\ \forall r\in\mathbb{Z}. \tag{9}$$

From (7) for $r=0,1$ and from (9) we obtain in particular

$$H^1_T(G,M')=H^2_T(G,M')=0 \tag{10}$$

thus we can apply (7) to M' and conclude

$$H^{-1}_T(G,M)=H^0_T(G,M')=0. \tag{11}$$

Replacing M by M' in the preceding argument we conclude

$$H^{-1}(G,M')=0, \tag{12}$$

but this is precisely saying

$$H^{-2}(G,M)=0. \tag{13}$$

We keep going. This proves

$$H^r_T(G,M)=0\ \forall r\in\mathbb{Z} \tag{14}$$

and proves the theorem when G is solvable.

We now address the general case. Recall that for $H\subset G$ an inclusion of finite groups and M a G-module, one has a restriction homomorphism

$$\mathrm{Res}: H^r_T(G,M)\to H^r_T(H,M). \tag{15}$$

For $r\geq 1$ this is the easy functor of restriction in cohomology. For $r\leq -2$ this comes from the more complicated functor in homology defined by

$$M_G\to M_H, x\mapsto \sum s^{-1}\cdot x \tag{16}$$

where s goes through a system of representatives of the set G/H. See [Har11, p.25, l.-8]. It is a good exercise to see that (16) is well defined! To go ahead, one needs the generalization of [Mil13, Cor.1.33] to Tate cohomology, which also follows from a generalization of [Mil13, Proposition 1.30]:

Claim 3 ([Har11], p.32, Ex. 4)**.** Let $G_p \subset G$ be a p-Sylow subgroup, then the restriction of $\mathrm{Res} : H^r_T(G, M) \to H^r_T(G_p, M)$ to the subgroup $H^r_T(G, M)$ of elements killed by multiplication by a p-power is injective.

As finite p-groups are solvable (see [MilGrp, Corollary 6.7] for a proof), applying (14) to G_p for all p we conclude that

$$\text{the torsion subgroup of } H^r_T(G, M) \text{ is } 0\ \forall r \in \mathbb{Z}. \tag{17}$$

Finally we know by the corollary [Mil13, Cor.1.31] of Shapiro's lemma that for $r > 0$, $|G|H^r_T(G, M) = 0$ for any G-module M. We conclude by (9) that

$$|G|H^r_T(G, M) = 0\ \forall r \in \mathbb{Z}. \tag{18}$$

Thus (17) together with (18) finish the proof. □

References

[Har11] Harari, D.: *Cohomologie galoisienne et théorie des nombres*, https://www.math.u-psud.fr/~harari/enseignement/cogal/poly.pdf

[Mil13] Milne, J.: *Class Field Theory*, http://www.jmilne.org/math/CourseNotes/CFT.pdf

[MilGrp] Milne, J.: *Group Theory*, http://www.jmilne.org/math/CourseNotes/GT.pdf