Proposition 1. *Let K be a complete discretely-valued nonarchimedean field, and let K' be a finite totally ramified extension of K which is separable. Let A and B be the discrete valuation rings in K and L, and let π_A and π_B be uniformizers in A and B. Then $A[\pi_B] = B$.*

Proof. In our proof, we will make use of the following lemma:

Lemma 1. *Let A be a PID with field of fractions K, and let B be the integral closure of A in a finite separable extension L of K of degree m. Let $\beta_1, \beta_2 \ldots \beta_m$ be a basis for L over K consisting entirely of elements of B, and let $d = disc(\beta_1, \beta_2 \ldots \beta_m)$. Then*

$$B \subset A\frac{\beta_1}{d} + A\frac{\beta_2}{d} + \ldots + A\frac{\beta_m}{d}$$

Proof. See Milne *Algebraic Number Theory* Proposition 2.43. □

It is clear that $A[\pi_B] \subset B$, we need to show that the reverse inclusion also holds. Assume that $[K' : K] = n$. Recall from the proof given in class, that $\{1, \pi_B, \ldots \pi_B^{n-1}\}$ are K linearly independent. Consider this integral basis of K' over K and let $d_0 = \mathrm{disc}(1, \pi_B, \ldots \pi_B^{n-1})$. Then $d_0 A = m_A^l$ for some $l \in \mathbb{N}$, thus $d_0 B = m_B^{ln}$, i.e. $d_0 = \pi_B^{ln} u$, with $u \in B^\times$. By our lemma, $d_0 B \subset A[\pi_B] \iff \pi_B^{ln} uB \subset A[\pi_B] \iff \pi_B^{ln} B \subset A[\pi_B]$. Therefore, we showed the existence of an $m_0 \in \mathbb{N}$ with

$$\pi_B^{m_0} B \subset A[\pi_B] \quad (1)$$

Now, because the extension is totally ramified, we have that $A/(\pi_A) \cong B/(\pi_B)$, which implies that the natural map $A \to B/(\pi_B)$ is surjective. Therefore $A + \pi_B B = B$, hence furthermore

$$A[\pi_B] + \pi_B B = B \quad (2)$$

Iterating (2) once, we obtain

$$\begin{aligned}
B &= A[\pi_B] + \pi_B B && (1)\\
&= A[\pi_B] + \pi_B \left(A[\pi_B] + \pi_B B\right) && (2)\\
&= \left(A[\pi_B] + \pi_B A[\pi_B]\right) + \pi_B^2 B && (3)\\
&= A[\pi_B] + \pi_B^2 B && (4)
\end{aligned}$$

Repeating this procedure, we obtain that in fact $B = A[\pi_B] + \pi_B^k B$ for every $k \in \mathbb{N}$, in particular it's true for $k = m_0$. Thus, using (1):

$$B = A[\pi_B] + \pi_B^{m_0} B \subset A[\pi_B]$$

and we are done.

□

Remark 1. *Once equality (2) is established, one can give an alternative proof. By our lemma it follows that $B/A[\pi_B]$ is a finitely generated A module, and thus one can apply Nakayama's Lemma in the form of Milne's Algebraic Number Theory lemma 1.9 b).*

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