NUMBER THEORY III – WINTERSEMESTER 2016/17

PROBLEM SET 3

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Exercise 1. Prove the following statements.

(a) On $\mathbb{R}$, every archimedean absolute value $|\cdot|$ such that $(\mathbb{R}, |\cdot|)$ is complete, is equivalent to the "usual" absolute value, defined for $x \in \mathbb{R}$ by

$$|x|_{\mathbb{R}} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{else} \end{cases}$$

(b) If $|\cdot|$ is an archimedean absoulte value on $\mathbb{C}$ extending $|\cdot|_{\mathbb{R}}$ then $|\cdot|$ is equivalent to the "usual" absolute value, defined for $x \in \mathbb{C}$ by

$$|x|_{\mathbb{C}} = |x\bar{x}|_{\mathbb{R}}^{1/2}.$$

Exercise 2. (a) Let K be a field complete with respect to a nonarchimedean absolute value $|\cdot|$. For $a_n \in K$, $n \in \mathbb{N}$, show that the series $\sum_{n\in\mathbb{N}} a_n$ converges in K if and only if $\lim_{n\to\infty} a_n = 0$.

(b) Let p be a prime number and let $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ with respect to the p-adic absolute value $|\cdot|_p = \frac{1}{p^{v_p(\cdot)}}$. Show that the series

$$\exp(x) := \sum_{n\geq 0} \frac{x^n}{n!},$$

converges if and only if $v_p(x) > 1/(p-1)$. (*Hint:* If $n = a_0 + a_1 p + \ldots + a_r p^r$ with $a_i \in \{0, \ldots, p-1\}$, compute that $v_p(n!) = \frac{n - \sum_{i=0}^r a_i}{p-1}$, then compare $\frac{\sum_{i=0}^r a_i}{p-1}$ with $\log(n)$.).

Definition. Let K be a field and let $|\cdot| : K \to \mathbb{R}_{\geq 0}$ be a function such that

(1) $|x| = 0$ if and only if $x = 0$.
(2) $|xy| = |x||y|$ for all $x, y \in K$.

The function $|\cdot|$ is called *weak absolute value*[1] if there exists a constant $C \in \mathbb{R}_{>0}$ such that the following axiom is satisfied:

(T_C) If $|x| \leq 1$, then $|1 + x| \leq C$.

Let $C_K := \inf\{C \in \mathbb{R}_{>0} | (T_C) \text{ is satisfied for } |\cdot|\}$.

Exercise 3. Let K be a field and let $|\cdot|$ be a weak absolute value. Prove the following statements.

(a) $C_K \geq 1$.

If you want your solutions to be corrected, please hand them in just before the lecture on November 8, 2016. If you have any questions concerning these exercises you can contact Lars Kindler via `kindler@math.fu-berlin.de` or come to Arnimallee 3, Office 109.

[1] This is nonstandard terminology.

(b) Show that $|\cdot|$ satisfies (T_{C_K}).

(c) For $C > 0$, the axiom (T_C) is satisfied if and only if for all $x, y \in K$
$$|x+y| \leq C \max\{|x|, |y|\}.$$

(d) $|\cdot|$ is an absolute value (i.e. it satisfies (1), (2) and the triangle inequality) if and only if $|\cdot|$ is a weak absolute value with $C_K \leq 2$. (*Hint*: If $|\cdot|$ is a weak absolute value with $C_K \leq 2$, show that for $n \leq 2^r < 2n$ and $x_1, \dots, x_n \in K$ we have
$$|x_1 + \ldots + x_n| \leq 2^r \max\{|x_1|, \dots, |x_n|\} \leq 2n \max\{|x_1|, \dots, |x_n|\} \leq 2n \sum_{i=1}^{n} |x_i|.$$
Use this to find a good upper bound for $|x+y|^n$ and let n go to infinity).

(e) A weak absolute value $|\cdot|$ is a non-archimedean absolute value if and only if $C_K = 1$.

Exercise 4. Let K be a field and $|\cdot|$ a weak absolute value. Prove the following statements, which show that sometimes weak absolute values are more convenient to work with than absolute values.

(a) If $|\cdot|$ satisfies the triangle inequality, show that for $t \in \mathbb{R}_{>0}$, $|\cdot|^t$ does not necessarily satisfy the triangle inequality.

(b) On the other hand, if $|\cdot|$ is a weak absolute value, and $t \in \mathbb{R}_{>0}$, then $x \mapsto |x|^t$ is a weak absolute value.

(c) For $t \in \mathbb{R}_{>0}$ define
$$C_K(t) = \inf\{C \in \mathbb{R}_{>0} \mid |\cdot|^t \text{ satisfies } (T_C)\}.$$
Then $C_K(t) = (C_K)^t$.

(d) Conclude that for every weak absolute value $|\cdot|$ there exists some $t \in \mathbb{R}_{>0}$ such that $|\cdot|^t$ is an absolute value.