NUMBER THEORY III – WINTERSEMESTER 2016/17

PROBLEM SET 14

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Let K be a field and L/K a finite Galois extension with Galois group G. Recall that a 2-cocycle for L/K is a map

$$\varphi : G \times G \to L^\times,$$

satisfying the *cocycle condition*

$$\rho(\varphi(\sigma,\tau)) \cdot \varphi(\rho,\sigma\tau) = \varphi(\rho\sigma,\tau) \cdot \varphi(\rho,\sigma).$$

for all $\rho, \sigma, \tau \in G$. To φ we attached a central simple algebra $A(\varphi)$ over K, containing L and split by L. If φ' is a second such 2-cocycle, we showed that $A(\varphi) \cong A(\varphi')$ if and only if there exists a map $\theta : G \to L^\times$, such that

$$\psi(\sigma,\tau)\varphi(\sigma,\tau)^{-1} = \theta(\sigma) \cdot \sigma(\theta(\tau)) \cdot \theta(\sigma\tau)^{-1},$$

for all $\sigma, \tau \in G$.

Exercise 1 (Inflation)**.** Let K be a field and let $L'/L/K$ be a tower of extensions such that L/K and L'/K are both finite Galois. Write $\mathrm{pr} : \mathrm{Gal}(L'/K) \twoheadrightarrow \mathrm{Gal}(L/K)$ for the restriction map.

(a) If $\varphi : \mathrm{Gal}(L/K)^2 \to L^\times$ is a 2-cocycle for L, let φ^* denote the composition

$$\varphi^* : \mathrm{Gal}(L'/K)^2 \xrightarrow{\mathrm{pr}^2} \mathrm{Gal}(L/K)^2 \xrightarrow{\varphi} L^\times \hookrightarrow L'^\times.$$

Show that $\varphi^* : \mathrm{Gal}(L'/K)^2 \to L'^\times$ is a 2-cocycle for L', and that this induces a homomorphism $H^2(\mathrm{Gal}(L/K), L^\times) \to H^2(\mathrm{Gal}(L'/K), L'^\times)$, the so called *inflation map* $\mathrm{infl}_L^{L'}$.

(b) We fix some notation:

- There is a homomorphism of L-algebras $L' \to \mathrm{End}_L(L')$, given by multiplication of L' on itself. After fixing an L-basis $\{b_1, \dots, b_r\}$ of L', we obtain an injective homomorphism of L-algebras $\lambda : L' \to M_r(L)$.
- For $\sigma \in \mathrm{Gal}(L'/K)$, write $\sigma(b_j) := \sum_{i=1}^r b_i m_{ij}(\sigma)$. Write $\mu(\sigma) := (m_{ij}) \in M_r(L)$. This defines a map

$$\mu : \mathrm{Gal}(L'/K) \to M_r(L),$$

which is not necessarily a homomorphism.
- If $M = (m_{ij}) \in M_r(L)$ and $\sigma \in \mathrm{Gal}(L/K)$, define $\sigma(M) := (\sigma(m_{ij}))$ to be the matrix in which σ has been applied to all entries. Show that for all $\sigma, \tau \in \mathrm{Gal}(L'/K)$,

$$\mu(\sigma\tau) = \mu(\sigma) \cdot \sigma(\mu(\tau))$$

and that for all $\gamma \in L'$,

$$\mu(\sigma) \cdot \lambda(\sigma(\gamma)) = \sigma(\lambda(\gamma)) \cdot \mu(\sigma).$$

If you want your solutions to be corrected, please hand them in just before the lecture on February 7, 2017. If you have any questions concerning these exercises you can contact Lars Kindler via `kindler@math.fu-berlin.de` or come to Arnimallee 3, Office 109.

- Let $\varphi : \mathrm{Gal}(L/K)^2 \to L^\times$ be a 2-cocycle. Consider the K-vector space $A(\varphi) \otimes_L M_r(L)$, where the left-$L$-structure of $A(\varphi)$ is used in the tensor product, i.e., for $\gamma \in L, x \in A(\varphi), y \in M_r(L)$, $\gamma(x \otimes y) = (\gamma x) \otimes y = x \otimes \gamma y$.
  Show that the map $x \otimes_K y \mapsto x \otimes_L y$ defines an isomorphism of K-vector spaces
$$A(\varphi) \otimes_K M_r(K) \xrightarrow{\cong} A(\varphi) \otimes_L M_r(L).$$
- The K-algebra structure on the left-hand-side gives rise to a K-algebra structure on the right hand side; using this structure and the embedding $\lambda : L' \hookrightarrow M_r(L)$, we consider L' as a subfield of $A(\varphi) \otimes_L M_r(L)$.
- Let $\varphi^* : \mathrm{Gal}(L'/K)^2 \to L'^\times$ be the 2-cocycle obtained by inflation. Denote by $\{e_\sigma | \sigma \in \mathrm{Gal}(L'/K)\}$ a basis of $A(\varphi^*)$, satisfying $e_\sigma e_\tau = \varphi^*(\sigma, \tau) e_{\sigma\tau}$ and $e_\sigma \gamma = \sigma(\gamma) e_\sigma$ for $\gamma \in L'$. Similarly, denote by $\{e_\tau | \tau \in \mathrm{Gal}(L/K)\}$ a basis of $A(\varphi)$ satisfying the analogous relations.
  Show that the map
$$A(\varphi^*) \to A(\varphi) \otimes_L M_r(L), \quad e_\sigma \mapsto e_{\sigma|_L} \otimes \mu(\sigma)$$
  is an isomorphism of K-algebras
- Conclude that there is a commutative diagram
$$\begin{array}{ccc} H^2(\mathrm{Gal}(L'/K), L'^\times) & \longrightarrow & \mathrm{Br}(K) \\ {\scriptstyle \mathrm{infl}_L^{L'}} \uparrow & \nearrow & \\ H^2(\mathrm{Gal}(L/K), L^\times). & & \end{array}$$

Exercise 2. Let K be a field and fix a separable closure $\overline{K}$ of K. Fix $a \in K^\times$. In this exercise we finally show that the map
$$X(K) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\overline{K}/K), \mathbb{Q}/\mathbb{Z}) \to \mathrm{Br}(K), \quad \chi \mapsto [A(\chi, a)]$$
is linear.

(a) Let $\chi, \chi' \in X(K)$ be characters. Let $K_\chi, K_{\chi'}, K_{\chi+\chi'}$ be the cyclic extensions of K defined by $\chi, \chi', \chi + \chi' \in X(K)$. Furthermore, let L be the compositum (in $\overline{K}$) of K_χ and $K_{\chi'}$. Show that $K_{\chi+\chi'} \subseteq L$ and that L/K is an abelian Galois extension.

Let $\varphi_{\chi,a}$, $\varphi_{\chi',a}$, $\varphi_{\chi+\chi',a}$ be the 2-cocycles attached to this data (see Problem Set 13, Ex. 3), and let $\varphi^*_{\chi,a}, \varphi^*_{\chi',a}, \varphi^*_{\chi+\chi',a}$ be the 2-cocycles $\mathrm{Gal}(L/K)^2 \to L^\times$, obtained by inflation.

(b) Show that $\varphi^*_{\chi+\chi',a}$ and $\varphi^*_{\chi,a} \cdot \varphi^*_{\chi',a}$ are equivalent 2-cocycles. This can be quite exhausting. To do it, you could proceed as follows:
- Let $m = \mathrm{ord}(\chi'), n = \mathrm{ord}(\chi)$.
- Let σ_0, σ_1 be the generators of $\mathrm{Gal}(K_\chi/K)$ and $\mathrm{Gal}(K_{\chi'}/K)$ determined by χ, χ'. Given $\tau, \tau' \in \mathrm{Gal}(L/K)$, write
$$\tau|_{K_\chi} = \sigma_0^{i_0}, \quad \tau'|_{K_\chi} = \sigma_0^{j_0},$$
$$\tau|_{K_{\chi'}} = \sigma_1^{i_1}, \quad \tau'|_{K_{\chi'}} = \sigma_1^{j_1},$$
  where $0 \leq i_0, j_0 < n$ and $0 \leq i_1, i_1 < m$.
  Then $\chi(\tau) + \chi'(\tau) = \frac{i_0}{n} + \frac{i_1}{m} = \frac{mi_0 + ni_1}{nm} \in \mathbb{Q}/\mathbb{Z}$. For natural numbers x, y write $[x]_y \in \{0, \ldots, y-1\}$ for the remainder which occurs when dividing x by y. Compute

that

$$\varphi^*_{\chi+\chi',a}(\tau,\tau') = \begin{cases} 1, & \text{if } \frac{[mi_0+ni_1]_{mn}}{mn} + \frac{[mj_0+nj_1]_{mn}}{mn} < 1 \\ a, & \text{if } \frac{[mi_0+ni_1]_{mn}}{mn} + \frac{[mj_0+nj_1]_{mn}}{mn} \geq 1 \end{cases}$$

- On the other hand, check that

$$(\varphi^*_{\chi,a}\varphi^*_{\chi',a})(\tau,\tau') = \begin{cases} 1, & \text{if } \frac{i_0+j_0}{n} < 1 \text{ and } \frac{i_1+j_1}{m} < 1 \\ a^2, & \text{if } \frac{i_0+j_0}{n} \geq 1 \text{ and } \frac{i_1+j_1}{m} \geq 1 \\ a, & \text{else} \end{cases}$$

- Define a map $\theta : \mathrm{Gal}(L/K) \to K^\times$ by

$$\theta(\tau) = \begin{cases} 1 & \text{if } \frac{i_0}{n} + \frac{i_1}{m} < 1 \\ a & \text{if } \frac{i_0}{n} + \frac{i_1}{m} \geq 1 \end{cases}$$

Define $d\theta : \mathrm{Gal}(L/K)^2 \to K^\times$, $d\theta(\tau,\tau') := \theta(\tau)\theta(\tau')\theta(\tau\tau')^{-1}$, and show that

$$\varphi^*_{\chi,a}\varphi^*_{\chi',a} = \varphi^*_{\chi+\chi',a}d\theta$$

(c) Conclude that

$$[A(\chi,a)] + [A(\chi',a)] = [A(\varphi^*_{\chi,a})] + [A(\varphi^*_{\chi',a})] = [A(\varphi^*_{\chi,a} \cdot \varphi^*_{\chi',a})] = [A(\varphi^*_{\chi+\chi',a})] = [A(\chi+\chi',a)].$$