NUMBER THEORY III – WINTERSEMESTER 2016/17

PROBLEM SET 12

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Exercise 1.

(a) Let K be a field and let D be a central division algebra over K. If L/K is a finite extension such that $D \otimes_K L \cong M_s(L)$ for some $s \geq 1$, then $s|[L : K]$.

(b) Let K be a local field of characteristic 0 and $n \in \mathbb{N}$. Fix an algebraic closure $\overline{K}$ and let $K_n \subseteq \overline{K}$ be the unique unramified subextension of degree n. Show that every class in $\mathrm{Br}(K_n/K)$ is represented by a division algebra D such that $\sqrt{\dim_k D}\,\big|\, n$.

Exercise 2.

(a) (*Newton's Lemma*) Let $(K, |\cdot|)$ be a complete discretely valued field and let A be its valuation ring with π a uniformizer. Given $f \in A[T]$, if $a_0 \in A$ is an element such that

$$|f(a_0)| < |f'(a_0)|^2$$

then there exists a root $a \in A$ of f such that

$$|a - a_0| \leq \frac{|f(a_0)|}{|f'(a_0)|^2}.$$

(b) If $(K, |\cdot|)$ is a local field of characteristic 0, equip $K^\times$ with the subspace topology of K. This makes $K^\times$ into a topological group. Show that for every $m \geq 1$, the subgroup $(K^\times)^m := \{x^m | x \in K^\times\}$ is open (*Hint: One can use Newton's Lemma to show that* $(\mathcal{O}_K^\times)^m := \{x^m | x \in \mathcal{O}_K^\times\}$ *contains a ball of a certain radius centered at* 1).

(c) Conclude that every subgroup $H \subseteq K^\times$ of finite index is open.

Exercise 3. Let K be a complete nonarchimedean field, fix a separable closure $\overline{K}$, and let $L \subseteq \overline{K}$ be a separable extension of K.

(a) Show that if $K' \subseteq \overline{K}$ is a finite unramified extension of K, then the compositum LK' of L and K' in $\overline{K}$ is unramified over L.

(b) Now assume in addition that the residue field of K is finite. Then for every n there exists a unique $K_n \subseteq \overline{K}$ such that K_n/K is unramified and has degree n. If $L \subseteq \overline{K}$ is any finite extension of K and if n is such that $K_n \cap L = K$, then show that L, K_n are linearly disjoint over K in $\overline{K}$, and that $L_n = LK_n \cong L \otimes_K K_n$ (this fills the gap in the lecture on Jan. 17).

If you want your solutions to be corrected, please hand them in just before the lecture on January 24, 2017. If you have any questions concerning these exercises you can contact Lars Kindler via `kindler@math.fu-berlin.de` or come to Arnimallee 3, Office 109.