NUMBER THEORY III – WINTERSEMESTER 2016/17

PROBLEM SET 11

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Exercise 1. (a) Let K be a field of characteristic $\neq 2$ and $a, b, c \in K^\times$. Show that $H(a, b; K) \cong H(ac^2, b; K) \cong H(a, bc^2; K)$, where $H(a, b; K)$ is the generalized quarternion algebra that we studied in the previous exercises.

(b) Show that in $\mathrm{Br}(\mathbb{R})$ the only nontrivial class is the class of $H(-1, -1; \mathbb{R})$, the "classical" quarternion algebra.

Exercise 2. In the lecture you sketched a proof of the following fact. Fill in the details. Let K be a field and fix an algebraic closure $\overline{K}$. For $a \in K^\times$ and for two continuous characters $\chi, \chi' : \mathrm{Gal}(\overline{K}/K) \to \mathbb{Q}/\mathbb{Z}$, show that

$$[A(\chi + \chi', a)] = [A(\chi, a)] + [A(\chi', a)]$$

in $\mathrm{Br}(K)$.

Exercise 3. Let R be a unital ring and for $n \in \mathbb{N}$ write $R_n := \mathrm{M}_n(R)$ for the ring of $n \times n$-matrices. Denote by Mod_R and Mod_{R_n} the categories of right modules over these rings. Let $e_{11} \in R_n$ be the matrix with zeros everywhere, except for a 1 in the top left entry. For $V \in \mathrm{Mod}_R$ write $G(V) := V^{\oplus n}$ and consider an element of $G(V)$ as a row vector of elements of V. Let R_n act on $G(V)$ from the right by matrix multiplication. For $U \in \mathrm{Mod}_{R_n}$ write $F(U) := V \cdot e_{11}$ and let R act on $F(U)$ on the right: for any $r \in R$, considered as a diagonal matrix in R_n, we have $Ve_{11}r = Vre_{11} \subseteq Ve_{11}$. Show that F, G define functors

$$G : \mathrm{Mod}_R \to \mathrm{Mod}_{R_n}, \quad F : \mathrm{Mod}_{R_n} \to \mathrm{Mod}_R$$

which are quasi-inverse equivalences, i.e., such that $FG \cong \mathrm{id}_{\mathrm{Mod}_R}$ and $GF \cong \mathrm{id}_{\mathrm{Mod}_{R_n}}$.

If you want your solutions to be corrected, please hand them in just before the lecture on January 17, 2017. If you have any questions concerning these exercises you can contact Lars Kindler via `kindler@math.fu-berlin.de` or come to Arnimallee 3, Office 109.