NUMBER THEORY III – WINTERSEMESTER 2016/17

PROBLEM SET 10

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Exercise 1. Let k be a field of characteristic $\neq 2$, and fix a separable closure $\bar{k}$ of k. Let $a, b \in k^\times$ and assume that $[k(\sqrt{a}) : k] = 2$. Let $\chi_a : \mathrm{Gal}(\bar{k}/k) \to \mathbb{Z}/2\mathbb{Z}$ be the unique surjective morphism corresponding to the quadratic extension $k(\sqrt{a})/k$. Show that the cyclic algebra $A(\chi_a, b)$ is isomorphic to the generalized quaternion algebra $H(a, b; k)$ that you know from the previous exercises.

Exercise 2. Let k be a field and fix a separable closure $\bar{k}$ of k. If $\chi : \mathrm{Gal}(\bar{k}/k) \to \mathbb{Q}/\mathbb{Z}$ is a continuous character and $a \in k^*$, then we write (χ, a) for the class of the cyclic algebra $A(\chi, a)$ in $\mathrm{Br}(k)$. In this exercise we prove that

$$(\chi, a) \cdot (\chi, b) = (\chi, ab)$$

for $a, b \in k^\times$. In particular, the class $(\chi, 1)$ is the neutral element of $\mathrm{Br}(k)$.

Let n be the order of χ. This means that the image of χ is $\frac{1}{n}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z}$. Let L be the fixed field of $\ker(\chi)$. Then χ defines an isomorphism $\mathrm{Gal}(L/k) \xrightarrow{\cong} \frac{1}{n}\mathbb{Z}/\mathbb{Z}$. Denote by $\sigma \in \mathrm{Gal}(L/k)$ the element corresponding to $\frac{1}{n} \mod \mathbb{Z}$. This is a generator of $\mathrm{Gal}(L/k)$.

Write $A = A(\chi, a)$, $B = A(\chi, b)$ and $C = A(\chi, ab)$. Then $L \subseteq A, B$ and considering A, B as left-L-modules, we define $V := A \otimes_L B$ (more concretely: $A \otimes_k B$ is unambiguous, as A, B are central, and $A \otimes_L B$ is the quotient of $A \otimes_k B$ by the relations $(\lambda x \otimes y - x \otimes \lambda y, \lambda \in L)$).

(a) Show/Convince yourself that V admits the structure of a right-$(A \otimes_k B)$-module, satisfying

$$(x \otimes_L y)(x' \otimes_k y') = (xx' \otimes_L yy')$$

for all $x, x' \in A$, $y, y' \in B$.

(b) Recall that there exists $\alpha \in A$ such that $A = \bigoplus_{i=0}^{n-1} L\alpha^i$ with $\alpha^n = a$, $\alpha^i \cdot \alpha^j = \alpha^{i+j}$ and $\alpha \cdot \lambda = \sigma(\lambda)\alpha$ for all $\lambda \in L$. Let $\beta \in B$ and $\gamma \in C$ denote analogous generators (i.e., $\beta^n = b$, $\gamma^n = ab$). Show/Convince yourself that V carries the structure of a left-C-module satisfying

$$\lambda\gamma^i \cdot (x \otimes_L y) = \lambda\alpha^i x \otimes_L \beta^i y$$

for all $\lambda \in L$, $x \in A$, $y \in B$, $i \in \{0, \ldots, n-1\}$.

(c) Show/Convince yourself that the two actions are compatible in the following sense:

$$\lambda\gamma \cdot ((x \otimes_L y) \cdot (x' \otimes_k y')) = (\lambda\gamma \cdot (x \otimes_L y)) \cdot (x' \otimes_k y')$$

for all $\lambda \in L$, $x, x' \in A$, $y, y' \in B$. Conclude that this yields a homomorphism $f : (A \otimes_k B)^{\mathrm{op}} \to \mathrm{End}_C(V)$.

(d) Show that f is an isomorphism of k-algebras. (*Hint: use that A, B are simple to show that f is injective. Since C is simple, you know the structure of V as a C-module. Compare dimensions to conclude that f is bijective*).

If you want your solutions to be corrected, please hand them in just before the lecture on January 10, 2017. If you have any questions concerning these exercises you can contact Lars Kindler via `kindler@math.fu-berlin.de` or come to Arnimallee 3, Office 109.

(e) Prove that $[(A \otimes_k B)^{\mathrm{op}}] = [C^{\mathrm{op}}]$ in $\mathrm{Br}(k)$ and conclude that $[A \otimes_k B] = [C]$ in $\mathrm{Br}(k)$.

Let A be a commutative ring and F an A-module. Consider the sequences of A-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0 \tag{1}$$

and

$$0 \to M_1 \otimes_A F \to M_2 \otimes_A F \to M_3 \otimes_A F \to 0 \tag{2}$$

- The A-module F is said to be *flat* if for every short exact sequence (1), the sequence (2) is also exact.
- The module F is said to be *faithfully flat*, if for every sequence (1), (1) is short exact *if and only if* (2) is short exact.
- If $\varphi : A \to B$ is a homomorphism of rings, then it can be shown that B is faithfully flat as an A-module if and only if B is a flat A-module and for every prime ideal $\mathfrak{p} \subseteq A$, there exists a prime ideal $\mathfrak{P} \subseteq B$ such that $\varphi^{-1}(\mathfrak{P}) = \mathfrak{p}$. We say that B is a *faithfully flat* A-algebra.

For the following fundamental theorem from commutative algebra, we have to fix some notation: If $\varphi : A \to B$ is a homomorphism of commutative rings, we write φ^* for the functor $- \otimes_A B$ from the category of A-modules to the category of B-modules given by tensoring with the A-algebra B given by φ.

We write $p_1, p_2 : B \to B \otimes_A B$, $p_1(b) = b \otimes 1$, $p_2(b) = 1 \otimes b$, and $p_{ij} : B \otimes_A B \to B \otimes_A B \otimes_A B$, $p_{12}(b \otimes b') = b \otimes b' \otimes 1$, $p_{13}(b \otimes b') = b \otimes 1 \otimes b'$, $p_{23}(b \otimes b') = 1 \otimes b \otimes b'$.

Theorem (Grothendieck's faithfully flat descent)**.** *Let $A \subseteq B$ be a faithfully flat extension of commutative rings. Let N be a B-module (or a B-algebra). There exists a bijection between the set of isomorphism classes of A-modules M (or A-algebras) such that $M \otimes_A B \cong N$ and the set of equivalence classes of $B \otimes_A B$-isomorphisms*

$$\varphi : p_1^* N = N \otimes_A B \cong B \otimes_A N = p_2^* N$$

satisfying the cocycle condition*, which is expressed as the commutativity of the following diagram:*

$$\begin{tikzcd}
p_{13}^* p_1^* N \arrow[r, equal] \arrow[d, "p_{13}^*\varphi"] & p_{12}^* p_1^* N \arrow[r, "p_{12}^*\varphi"] & p_{12}^* p_2^* N \arrow[r, equal] & p_{23}^* p_1^* N \arrow[d, "p_{23}^*\varphi"] \\
p_{13}^* p_2^* N \arrow[rrr, equal] & & & p_{23}^* p_2^* N
\end{tikzcd}$$

An isomorphism φ satisfying the cocycle condition is called descent datum *for N relative to $A \subseteq B$. Finally, two $B \otimes_A B$-isomorphisms $\varphi, \psi : p_1^* N \to p_2^* N$ are said to be* equivalent *if there exists an automorphism $\mu : N \xrightarrow{\cong} N$, such that $\varphi = p_1^* \mu \circ \psi \circ p_2^* \mu^{-1}$*

Exercise 3. Let L/K be a finite Galois extension with Galois group G. Use the descent theorem to show that (up to isomorphism) giving a central simple algebra A over K is equivalent to giving a central simple algebra B over L together with a *descent datum*, that is together with an isomorphism $B \otimes_K L \xrightarrow{\cong} L \otimes_K B$ of $L \otimes_K L$-algebras satisfying the cocycle condition.